Bubbles and Credit Constraints[*]

Jianjun Miao[†] Pengfei Wang[‡]

July 5, 2013

Abstract

We provide an infinite-horizon model of a production economy with credit-driven stock-price bubbles, in which firms meet stochastic investment opportunities and face endogenous credit constraints. Firms have limited commitment to repay debt. Credit constraints ensure that default never occurs in equilibrium. We show that bubbles in firm value can exist to relax credit constraints and improve investment efficiency. We provide conditions under which bubbles can coexist with other types of assets. We show that the collapse of bubbles leads to a recession and a stock market crash. There is a government policy that can eliminate bubbles and achieve efficient allocation.

Keywords: Credit-Driven Bubbles, Credit Constraints, Asset Price, Arbitrage, Q Theory, Liquidity, Multiple Equilibria

JEL codes: E2, E44

[*]We thank Bruno Biais, Toni Braun, Markus Brunnermeier, Henry Cao, Christophe Chamley, Tim Cogley, Russell Cooper, Douglas Gale, Mark Gertler, Simon Gilchrist, Christian Hellwig, Hugo Hopenhayn, Andreas Hornstein, Boyan Jovanovic, Bob King, Nobu Kiyotaki, Anton Korinek, Felix Kubler, Kevin Lansing, John Leahy, Eric Leeper, Zheng Liu, Gustavo Manso, Ramon Marimon, Erwan Morellec, Fabrizio Perri, Jean-Charles Rochet, Tom Sargent, Jean Tirole, Jon Willis, Mike Woodford, Tao Zha, Lin Zhang, and, especially, Wei Xiong and Yi Wen for helpful discussions. We have also benefitted from comments from seminar or conference participants at the BU macro lunch workshop, Cheung Kong Graduate School of Business, European University Institute, Indiana University, New York University, Toulouse School of Economics, University of Lausanne, University of Southern Denmark, University of Zurich, Zhejiang University, the 2011 Econometric Society Summer Meeting, the 2011 International Workshop of Macroeconomics and Financial Economics at the Southwestern University of Finance and Economics, the Federal Reserve Banks of Atlanta, Boston, San Francisco, Richmond, and Kansas, the Theory Workshop on Corporate Finance and Financial Markets at Stanford, Shanghai University of Finance and Economics, the 2011 SED conference in Ghent, and the 7th Chinese Finance Annual Meeting. First version: December 2010.

[†]Department of Economics, Boston University, 270 Bay State Road, Boston, MA 02215. Tel.: 617-353-6675. Email: miaoj@bu.edu. Homepage: http://people.bu.edu/miaoj.

[‡]Department of Economics, Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong. Tel: (+852) 2358 7612. Email: pfwang@ust.hk

1 Introduction

This paper provides a theory of credit-driven stock market bubbles. Our theory is motivated by two observations. First, the United States has experienced stock market booms and busts, which is difficult to be explained entirely by fundamentals. Second, stock market booms are often accompanied by credit market booms. This suggests that one possible cause of stock market bubbles is excessive liquidity in the financial system, inducing lax or inappropriate bank lending standards.[1] These two observations are illustrated in Figure 1, which plots the U.S. historical data of the monthly price-earnings ratio and the credit spread. This figure shows that the stock market is highly volatile relative to fundamentals. In addition, there is a positive comovement between the stock market and the credit market, given that the credit spread is a good indicator of the credit market conditions (Gilchrist, Yankov, and Zakrajsek (2009)).

The above two observations are seen in many other countries, especially in emerging market countries. For example, overoptimism in the 1990s towards an "East Asian miracle" generated high economic growth in East Asian countries. Capital account and financial market liberalization contributed to large capital inflows and generated a lending boom. The rapid increase in asset prices including housing prices and stock prices were accompanied by a large expansion of domestic credit through under-regulated banking systems (Collyns and Senhadji (2002)).

To formalize our theory, we construct a tractable model of a production economy in which households are infinitely lived and trade firm stocks and household bonds (e.g., private IOUs). There is no aggregate uncertainty. Households are risk neutral so that the rate of return on any traded asset is equal to the constant subjective discount rate.[2] A continuum of firms meet idiosyncratic stochastic investment opportunities as in Kiyotaki and Moore (1997, 2005, 2008). In the baseline model, we suppose that firms can use internal funds and external borrowing, but not other sources of funding, to finance investment. Firms face endogenous credit constraints, which are modeled in a similar way to Bulow and Rogoff (1989), Kehoe and Levine (1993), Kiyotaki and Moore (1997), Alvarez and Jermann (2000), Albuquerque and

[1] For example, Axel A. Weber, the former president of the Deutsche Bundesbank, has argued that "The past has shown that an overly generous provision of liquidity in global financial markets in connection with a very low level of interest rates promotes the formation of asset-price bubbles." (http://www.bloomberg.com/apps/news?pid=newsarchive&sid=a5S5Boes29lo)

[2] These two assumptions are adopted for simplicity. Miao and Wang (2011) introduce a concave utility function to study sectoral bubbles and endogenous growth. Miao, Wang, and Xu (2012b) study stock market bubbles and business cycles in a DSGE model with risk-averse households and multiple sources of aggregate shocks.

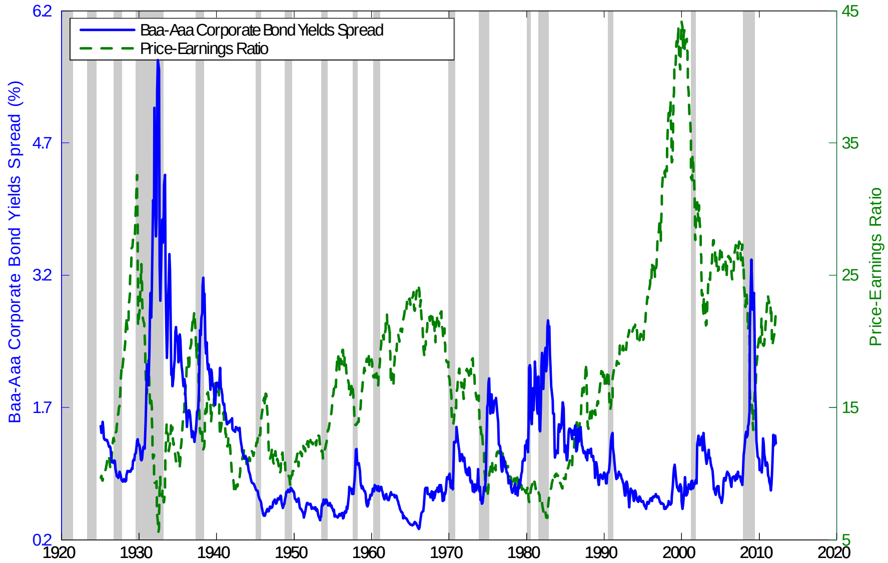

Figure 1: The credit spread and the price-earnings ratio. Shaded areas represent NBER recession dates. Source: Board of Governors of the Federal Reserve System and Robert Shiller's website: http://www.econ.yale.edu/~shiller/data.htm.

Hopenhayn (2004), and Jermann and Quadrini (2012). The key idea is that borrowers (firms) have limited commitment and debt repayments may not be fully enforced.

We consider two mechanisms to enforce debt repayments. First, a firm must pledge its physical assets (capital) as collateral. If the firm does not repay its debt, then it loses its collateralized assets and the right to run the firm all to the lender.[3] Thus, the collateral value to the lender is equal to the market value of the firm with the collateralized assets. The lender and the firm will renegotiate the debt such that the debt is limited by this collateral value. We call the resulting credit constraint the collateral constraint. In the baseline model, we focus on this type of credit constraint.

Unlike Kiyotaki and Moore (1997) who assume that the market value of the collateral is equal to the liquidation value of the collateralized assets, we assume that it is equal to the *going-concern value* of the reorganized firm with these assets. Because the going-concern value is priced in the stock market, it may contain a bubble component. If both the lenders and the credit-constrained borrowers (firms in our model) optimistically believe that the collateral value is high possibly because of bubbles, the borrowers will want to borrow more and the lenders

[3]In Section 7.2, we study the case where a fraction of firm value is used as collateral directly. This does not change our key insights and results.

will not mind lending more. Consequently, firms can finance more investment and accumulate more assets for future production, making their assets indeed more valuable.[4] This positive feedback loop mechanism makes the lenders' and the borrowers' beliefs self-fulfilling and allows bubbles to exist in equilibrium. We refer to this equilibrium as the bubbly equilibrium.

The second mechanism of enforcing debt repayments is for the firm to be subject to a penalty if it does not repay its debt. There is no collateral to guarantee the repayment of debt. In the event of a default, the firm is excluded from the financial market forever. In a no-default equilibrium, the continuation value of the firm must be at least as large as the outside value on default, which is the autarky value when the firm uses only internal funds to finance investment. We show that no bubble can exist for the firm after it defaults. We call the resulting credit constraint the self-enforcing constraint. This constraint effectively imposes an endogenous debt limit. We show that a bubble can also exist through the positive feedback loop mechanism discussed earlier. Specifically, if people believe that the value of a no-default firm contains a bubble, then the bubble helps relax the credit constraint because it reduces the incentive to default. As a result, the firm can borrow more to finance more investment. The increased investment causes firm value to go up, justifying initial optimistic beliefs. We show that the bubbly equilibrium with self-enforcing debt constraints is a special case of the bubbly equilibrium in the baseline model with collateral constraints in which an empty firm with zero assets is pledged as collateral or a bubble is effectively pledged as collateral. This result is reminiscent to the modeling of credit constraints in Martin and Ventura (2011, 2012).

Of course, there is another equilibrium in which no one believes in bubbles and hence bubbles do not appear. We call this equilibrium the bubbleless equilibrium. We provide explicit conditions to determine which type of equilibrium can exist. We prove that the economy has two steady states: a bubbly one and a bubbleless one. Both steady states are inefficient due to credit constraints and both are local saddle points. Thus, multiple equilibria in our model are not generated by indeterminacy as in the literature surveyed by Benhabib and Farmer (1999) and Farmer (1999). We show that the stable manifold is one-dimensional for the bubbly steady state, but two-dimensional for the bubbleless steady state. On the former stable manifold,

[4] Using firm-level data during the asset price bubble in Japan in the late 1980s, Goyal and Yamada (2004) find that investment responds significantly to stock price bubbles. Using a source of exogenous variation in collateral value provided by the property market collapse in Japan in the early 1990s, Gan (2007) finds a large impact of collateral on the corporate investments of a large sample of Japanese manufacturing firms. She shows that for every 10 percent drop in collateral value, the investment rate of an average firm is reduced by 0.8 percentage point. Chaney, Sraer, and Thesmar (2009) document similar evidence for the US economy during the 1993-2007 period.

bubbles persist in the steady state. But on the latter stable manifold, bubbles eventually burst.[5]

As Tirole (1982) and Santos and Woodford (1997) point out, it is difficult to generate rational bubbles for economies with infinitely lived agents. The intuition is as follows. A necessary condition for bubbles to exist is that the growth rate of bubbles cannot exceed the growth rate of the economy. Otherwise, investors cannot afford to buy into bubbles. In a deterministic economy, bubbles on assets with exogenous payoffs or on intrinsically useless assets must grow at the interest rate by the no-arbitrage principle. Thus, the interest rate cannot exceed the growth rate of the economy. This implies that the present value of aggregate endowments must be infinity. In an overlapping generations economy, this condition implies that the bubbleless equilibrium must be dynamically inefficient (see Tirole (1985)).

In our model, the growth rate of the economy is zero and the interest rate on household bonds is positive. In addition, the bubbleless equilibrium is dynamically efficient. But how do we reconcile our result with that in Santos and Woodford (1997) or Tirole (1985)? The key is that the bubbles in our model are attached to productive assets (capital) with endogenous payoffs. A distinguishing feature of our model is that bubbles in firm value have real effects and affect firm dividends. Although a no-arbitrage equation for these bubbles still holds in that the rate of return on bubbles is equal to the interest rate on household bonds, the growth rate of bubbles is not equal to this rate. Rather, it is equal to the interest rate minus the "collateral yield." The collateral yield comes from the fact that bubbles help relax the collateral constraints and allow firms to make more investment.

We extend our analysis to include other types of assets such as intertemporal corporate bonds, assets with rents (e.g., tree), and assets without rents (pure bubble, e.g., tulip). Suppose that firms can trade one of these assets to finance investment. We study the conditions under which firm bubbles (i.e., the firm's stock price bubbles) can coexist with other types of assets. If an asset can play the same role as a firm bubble in helping firms finance investment, then this asset will generate additional dividends to the firms, which are identical to the collateral yield. If, in addition, this asset delivers positive rents, then it dominates a bubble and hence they cannot coexist in equilibrium. But if this asset is a pure bubble, then it is a perfect substitute

[5]In Chapter 14 of Tirole's (2006) textbook, he shows that there may exist multiple equilibria in a simplified variant of the Kiyotaki and Moore (1997) model. In contrast to ours, these equilibria are characterized by a one-dimensional nonlinear dynamical system. Some equilibria may exhibit cycles. The steady states of these equilibria are not saddle points, unlike in our paper. We would like to thank Jean Tirole for a helpful discussion on this point.

for the firm bubble. Only the total size of the bubble can be determined in equilibrium. For a firm bubble to coexist with a corporate bond, the equilibrium interest rate on the corporate bond must be zero in the steady state. In addition, we need to introduce market frictions such as short-sales constraints (e.g., Kocherlakota (1992, 2009)). Without market frictions, the economy would achieve the efficient equilibrium and no bubble would exist.

We also show that if there is economic growth, then corporate bonds with a positive interest rate can coexist with firm bubbles, and assets with positive dividends can also coexist with firm bubbles provided the dividends grow at a rate lower than the rate of economic growth.

So far, we have only considered deterministic bubbles. Following Blanchard and Watson (1982) and Weil (1987), we construct a third type of equilibrium with stochastic bubbles in the baseline model. In this equilibrium, households believe that there is a positive probability that bubbles will burst at each date. When bubbles burst, they cannot reappear. We show that when all economic agents believe that the probability that a bubble will burst is small enough, an equilibrium with stochastic bubbles exists. In contrast to Weil (1987), we show that after a bubble bursts, a recession occurs in that there is a credit crunch and consumption and output fall eventually. In addition, immediately after the bubble bursts, investment falls discontinuously and the stock market crashes, i.e., the stock price also falls discontinuously. Note that the recession and the stock market crash occur without any exogenous shock to the fundamentals of the economy.

What is an appropriate government policy in the wake of a bubble collapse? The inefficiency in our model comes from the firms' credit constraints. The collapse of bubbles tightens these constraints and impairs investment efficiency. To overcome this inefficiency, the government may issue public bonds backed by lump-sum taxes. Both households and firms can trade these bonds subject to short-sales constraints. Public bonds serve as a store of value to households and firms and also as collateral to firms. Thus, public assets can help relax collateral constraints and play the same role as bubbles do. They dominate firm bubbles. The government constantly retires public bonds at the interest rate to maintain a constant total bond value and pays the interest payments of these bonds by levying lump-sum taxes. We show that this policy allows the economy to achieve the efficient equilibrium.

Some papers in the literature (e.g., Scheinkman and Weiss (1986), Kocherlakota (1992, 2008), Santos and Woodford (1997) and Hellwig and Lorenzoni (2009)) also find that infinite-horizon models with borrowing constraints may generate rational bubbles. Unlike these papers which study pure exchange economies, our paper analyzes a production economy. As mentioned above, our paper differs from these and the papers cited below in that we focus on bubbles in stock prices whose payoffs are endogenously determined by investment and affected by bubbles.[6]

Our paper is closely related to Caballero and Krishnamurthy (2006), Kocherlakota (2009), Wang and Wen (2012), Farhi and Tirole (2012), and Martin and Ventura (2011, 2012). Like our paper, these papers contain the idea that bubbles can help relax borrowing constraints and improve investment efficiency. Building on Kiyotaki and Moore (2008), Kocherlakota (2009) studies an economy with infinitely lived entrepreneurs. These entrepreneurs meet stochastic investment opportunities and are subject to collateral constraints. Land is used as the collateral. Unlike Kiyotaki and Moore (1997) or our paper, Kocherlakota (2009) assumes that land is intrinsically useless (i.e. it has no rents or dividends) and cannot be used as input for production. Wang and Wen (2012) provide a model similar to that in Kocherlakota (2009). They study asset price volatility and bubbles that may grow on assets with exogenous rents. They assume that these assets cannot be used as input for production. Our model can also generate bubbles on intrinsically useless assets as long as these assets can be used to finance investment and households face short-sales constraints. The latter assumption is standard in the literature (e.g., Kocherlakota (1992, 2009) and Wang and Wen (2012)). We show that bubbles on intrinsically useless assets and bubbles in firm value are perfect substitutes in our model.

Building on Diamond (1965) and Tirole (1985), Caballero and Krishnamurthy (2006), Farhi and Tirole (2012), and Martin and Ventura (2011, 2012) study bubbles in overlapping generations models with credit constraints. Caballero and Krishnamurthy (2006) show that stochastic bubbles are beneficial because they provide domestic stores of value, thereby reducing capital outflows while increasing investment. But they come at a cost, as they expose the country to bubble crashes and capital flow reversals. Farhi and Tirole (2012) assume that entrepreneurs may use bubbles and outside liquidity to relax the credit constraints. They study the interplay between inside and outside liquidity. Martin and Ventura (2011, 2012) use a model with bubbles to shed light on the recent financial crisis.

Our discussion of government policy is related to that in Caballero and Krishnamurthy (2006) and Kocherlakota (2009). As in these studies, government bonds can serve as collateral to help relax credit constraints in our model. Unlike their proposed policies, our proposed

[6] See Scheinkman and Xiong (2003) and Burnside, Eichenbaum and Rebelo (2011) for models of bubbles based on heterogeneous beliefs. See Shiller (2005) for a theory of bubbles based on irrational exuberance. See Brunnermeier (2009) for a survey of various theories of bubbles. See Xiong and Yu (2011) and Jovanovic (2013) for recent empirical evidence of asset price bubbles.

policy requires that government bonds be backed by lump-sum taxes and can make the economy achieve the efficient allocation. Unbacked public assets are intrinsically useless and may have positive value (a bubble) if households face short-sales constraints. Issuing unbacked public assets can boost the economy after the collapse of stock price bubbles. But the real allocation is still inefficient and the bubble on unbacked public assets can also burst. After bursting, the economy enters a recession again.

The rest of the paper is organized as follows. Section 2 presents a baseline model with collateral constraints. Section 3 derives the equilibrium system. Section 4 analyzes the bubbleless equilibrium, while Section 5 analyzes the bubbly equilibrium. Section 6 introduces several types of assets to the baseline model. Section 7 analyzes a model in which firm value is used as collateral and another model with self-enforcing debt constraints. Section 8 studies stochastic bubbles and government policy. Section 9 concludes. Appendix A contains proofs of results in the main text. Appendix B provides technical details for Section 7.

2 The Baseline Model

We consider an infinite-horizon production economy. There is no aggregate uncertainty. Time is denoted by $t = 0, dt, 2dt, 3dt, ...$. The length of a time period is dt. For analytical convenience, we shall take the limit of this discrete-time economy as dt goes to zero when characterizing equilibrium dynamics. The continuous-time limit is more convenient for analyzing local dynamics around a steady state. But the discrete-time setup helps us better understand intuition. We thus start with the model with discrete-time approximations heuristically, and relegate the continuous-time formulation to the appendices.

2.1 Households

There is a continuum of identical households of unit mass. Each household is risk neutral and derives utility from a consumption stream $\{C_t\}$ according to the following utility function:

$$\sum_{t\in\{0,dt,2dt,...\}} e^{-rt} C_t dt,$$

where r is the subjective rate of time preference.[7] Households supply labor inelastically. The labor supply is normalized to one. Households trade firm stocks and risk-free household bonds

[7] Introducing a general concave utility function allows us to endogenize interest rate, but it makes analysis more complex. It will not change our key insights though (see Miao and Wang (2011) and Miao, Wang and Xu (2012b)).

(e.g., private IOUs). The net supply of household bonds is zero and the net supply of any stock is one. Because there is no aggregate uncertainty, r is equal to the interest rate on the household bonds and also to the rate of return on each stock. In Section 7.2, we introduce other types of assets in the economy. No arbitrage implies that the rate of return on any traded asset must be equal to r. Households may deposit funds in financial intermediaries which make loans to firms without frictions.

2.2 Firms

There is a continuum of firms of unit mass. There is no entry or exit. Firms are indexed by $j \in [0,1]$. Each firm j combines labor N_t^j and capital K_t^j to produce output according to the following Cobb-Douglas production function:

$$Y_t^j = (K_t^j)^\alpha (N_t^j)^{1-\alpha}, \quad \alpha \in (0,1).$$

After solving the static labor choice problem, we obtain the operating profits

$$R_t K_t^j = \max_{N_t^j} \ (K_t^j)^\alpha (N_t^j)^{1-\alpha} - w_t N_t^j, \tag{1}$$

where w_t is the wage rate and R_t is given by

$$R_t = \alpha \left(\frac{w_t}{1-\alpha} \right)^{\frac{\alpha-1}{\alpha}}. \tag{2}$$

We will show later that R_t is equal to the marginal product of capital in equilibrium.

Following Kiyotaki and Moore (1997, 2005, 2008), we assume that each firm j meets an opportunity to invest in capital with probability πdt at time t. With probability $1 - \pi dt$, no investment opportunity arrives. Thus, capital evolves according to

$$K_{t+dt}^j = \begin{cases} (1-\delta dt) K_t^j + I_t^j & \text{with probability } \pi dt \\ (1-\delta dt) K_t^j & \text{with probability } 1 - \pi dt \end{cases}, \tag{3}$$

where $\delta > 0$ is the depreciation rate of capital and I_t^j is the investment level. This assumption captures firm-level investment lumpiness and generates ex post firm heterogeneity. Assume that the arrival of an investment opportunity is independent across firms and over time. In Section 7.2, we introduce idiosyncratic investment efficiency shocks with a continuous distribution. This modeling does not change our key insights. In a model without idiosyncratic investment shocks, Miao and Wang (2012a) introduce idiosyncratic productivity shocks and show that credit-driven bubbles can still emerge. These bubbles help raise total factor productivity endogenously.

Let the ex ante market value of the firm prior to the realization of an investment opportunity shock be $V_t(K_t^j)$, where we suppress aggregate state variables in the argument. It satisfies the following Bellman equation:

$$\begin{aligned} V_t(K_t^j) &= \max_{I_t^j, L_t^j} \pi dt \left\{ D_t^j - L_t^j + e^{-rdt} V_{t+dt}((1-\delta dt) K_t^j + I_t^j) \right\} \\ &\quad + (1-\pi dt) \left\{ R_t K_t^j dt + e^{-rdt} V_{t+dt}((1-\delta dt) K_t^j) \right\}, \end{aligned} \tag{4}$$

subject to the flow-of-funds constraints

$$D_t^j + I_t^j = R_t K_t^j dt + L_t^j,$$

and some constraints on investment to be specified shortly. Here, D_t^j represents dividends (new equity) if $D_t^j \geq (<)\, 0$, $R_t K_t^j dt$ represents internal funds, and L_t^j represents loans from financial intermediaries.[8] As will be shown in Section 3, the optimization problem in (4) is not well defined if there is no constraint on investment given our assumption of a constant-returns-to-scale technology. Thus, we impose an upper bound and a lower bound on investment. For the lower bound, we assume that investment is irreversible in that $I_t^j \geq 0$. It turns out that this constraint will never bind in our analysis below. For the upper bound, we assume that investment is financed by internal funds and external borrowing. We also assume that external equity is so costly that no firms would raise new equity to finance investment.[9] This means that $D_t^j \geq 0$. Without a constraint on new equity issuance, firms can overcome financial constraints and a credit-driven bubble cannot exist.

We now write the investment constraint as

$$0 \leq I_t^j \leq R_t K_t^j dt + L_t^j. \tag{5}$$

In the continuous-time limit as $dt \to 0$, we have $0 \leq I_t^j \leq L_t^j$. For simplicity, we consider intratemporal loans as in Carlstrom and Fuerst (1997) and Jermann and Quadrini (2012). These loans are taken at the beginning of the period and repaid at the end of the period

[8] Note that an invesment opportunity arrives at a Poisson rate. Thus, investment, loans, and dividend payments are lumpy, but earnings $R_t K_t^j dt$ arrive continuously as a flow.

[9] This assumption reflects the fact that external equity financing is more costly than debt financing. Bernanke et al. (1999), Carsltrom and Fuerst (1997), and Kiyotaki and Moore (1997) make the same assumption. We can relax this assumption by allowing firms to raise a limited amount of new equity. For example, we may replace (5) with

$$0 \leq I_t^j \leq R_t K_t^j dt + \mu K_t^j + L_t^j,$$

where μK_t^j represents the upper bound of new equity. In this case, our analysis and insights still hold with small modifications.

by selling firm equity. They do not incur interests.[10] After making investment, the firm's market value is $e^{-rdt}V_{t+dt}((1-\delta dt)\,K_t^j+I_t^j)$. The loan repayment is financed out of this value. Specifically, the firm can sell a fraction s_t^j of equity shares to the market at the end of period t, where $s_t^j = L_t^j / \left[e^{-rdt}V_{t+dt}((1-\delta dt)\,K_t^j+I_t^j)\right]$. It then uses the proceeds to repay loans.

The key assumption of our model is that loans are subject to credit constraints. In the baseline model, we consider the following collateral constraint:

$$L_t^j \le e^{-rdt}V_{t+dt}(\xi K_t^j). \tag{6}$$

We interpret this constraint as an incentive constraint in an optimal contract between firm j and the lender with limited commitment: Given a history of information at date t, in the time interval $[t, t+dt]$, the contract specifies investments I_t^j and loans L_t^j at the beginning of period t, and repayments L_t^j at the end of period $t+dt$, only when an investment opportunity arrives with Poisson probability πdt. When no investment opportunity arrives, the firm does not invest and hence does not borrow. Firm j may default on debt at the end of period t. If it defaults, then the firm and the lender will renegotiate the loan repayment. In addition, the lender has the right to reorganize the firm. Because of default costs, the lender can only seize a fraction ξ of capital K_t^j. Alternatively, we may interpret ξ as an efficiency parameter in that the lender may not be able to efficiently use the firm's assets K_t^j. The lender can run the firm with these assets at the beginning of period $t+dt$ and obtain firm value $e^{-rdt}V_{t+dt}(\xi K_t^j)$. Or it can sell these assets to a third party at the going-concern value $e^{-rdt}V_{t+dt}(\xi K_t^j)$ if the third party can run the firm using assets ξK_t^j at the beginning of period $t+dt$. This value is the threat value (or the collateral value) to the lender at the end of period t. Following Jermann and Quadrini (2012), we assume that the firm has all the bargaining power in the renegotiation and the lender obtains only the threat value. The key difference between our modeling and that of Jermann and Quadrini (2012) is that the threat value to the lender is the going-concern value in our model, while Jermann and Quadrini (2012) assume that the lender liquidates the firm's assets and obtains the liquidation value in the event of default.[11]

Enforcement requires that, when an investment opportunity arrives at date t, the continuation value to the firm of not defaulting be no smaller than the continuation value of defaulting,

[10] In Section 6.1, we incorporate intertemporal bonds and allow firms to save. We show that our key insights and analysis carry over to this setup.

[11] U.S. bankruptcy law has recognized the need to preserve the going-concern value when reorganizing businesses in order to maximize recoveries by creditors and shareholders (see 11 U.S.C. 1101 et seq.). Bankruptcy laws seek to preserve going concern value whenever possible by promoting the reorganization, as opposed to the liquidation, of businesses.

that is,

$$\begin{aligned} & e^{-rdt}V_{t+dt}((1-\delta dt)\,K_t^j + I_t^j) - L_t^j \\ \geq \;& e^{-rdt}V_{t+dt}((1-\delta dt)\,K_t^j + I_t^j) - e^{-rdt}V_{t+dt}(\xi K_t^j). \end{aligned} \tag{7}$$

This incentive constraint is equivalent to the collateral constraint in (6). This constraint ensures that there is no default in an optimal contract. The firm repays loans using funds from sales of its equity and hence its market value is reduced by L_t^j.

In the continuous-time limit, the collateral constraint becomes

$$L_t^j \leq V_t(\xi K_t^j). \tag{8}$$

Note that our modeling of the collateral constraint is different from that of Kiyotaki and Moore (1997). We may write the Kiyotaki-Moore-type collateral constraint in our continuous-time framework as

$$L_t^j \leq \xi Q_t K_t^j, \tag{9}$$

where Q_t represents the shadow price of capital. The expression $\xi Q_t K_t^j$ is the shadow value of the collateralized assets or the liquidation value.[12] In Section 5, we shall argue that this type of collateral constraint will rule out bubbles. By contrast, according to (6), we allow the collateralized assets to be valued in the stock market as the going-concern value when the firm is reorganized and kept running using the collateralized assets after default. If both the firm and the lender believe that the firm's assets are overvalued due to stock market bubbles, then these bubbles will help relax the collateral constraint, providing a positive feedback loop mechanism.

2.3 Competitive Equilibrium

Let $K_t = \int_0^1 K_t^j dj$, I_t, $N_t = \int_0^1 N_t^j dj$, and $Y_t = \int_0^1 Y_t^j dj$ denote the aggregate capital stock, average investment of firms with investment opportunities, the aggregate labor demand, and aggregate output, respectively. Then a competitive equilibrium is defined as sequences of $\{Y_t\}$, $\{C_t\}$, $\{K_t\}$, $\{I_t\}$, $\{N_t\}$, $\{w_t\}$, $\{R_t\}$, $\{V_t(K_t^j)\}$, $\{I_t^j\}$, $\{K_t^j\}$, $\{N_t^j\}$ and $\{L_t^j\}$ such that

[12] Note that our model differs from the Kiyotaki and Moore model in market arrangements, besides other specific modeling details. Kiyotaki and Moore assume that there is a market for physical capital (corresponding to land in their model), but there is no stock market for trading firm shares. In addition, they assume that households and entrepreneurs own firms and trade physical capital in the capital market. By contrast, we assume that households trade firm shares in the stock market and that firms own physical capital and make investment.

households and firms optimize and markets clear in that:

$$
\begin{aligned}
N_t &= 1, \\
C_t + \pi I_t &= Y_t, \\
K_{t+dt} &= (1 - \delta dt) K_t + \pi I_t dt.
\end{aligned}
$$

3 Equilibrium System

We first solve an individual firm's optimal contract problem (4) subject to (3), (5), and (6) when the wage rate w_t or R_t in (2) is taken as given. This problem does not give a contraction mapping and hence may admit multiple solutions. We conjecture that the ex ante firm value takes the following form:

$$
V_t(K_t^j) = v_t K_t^j + b_t, \tag{10}
$$

where v_t and b_t are to be determined and depend on aggregate states only. Note that $b_t = 0$ is a possible solution. In this case, we may interpret $v_t K_t^j$ as the fundamental value of the firm. The fundamental value is proportional to the firm's assets K_t^j, which has the same form as that in Hayashi (1982). Intuitively, the firm has no fundamental value if it has no assets ($K_t^j = 0$). There may be another solution in which $b_t > 0$ is generated from optimistic beliefs.[13] In this case, we interpret b_t as a bubble since the firm is still valued at b_t even when there is no market fundamental, i.e., $K_t^j = 0$. In Section 7.2, we show that when an intrinsically useless asset is traded in the market, its price and b_t follow the same asset pricing equation (i.e., they are perfect substitutes), further justifying our interpretation of b_t as a bubble. Alternatively, one can interpret b_t as a sunspot or speculative component in firm value, which depends on people's beliefs.[14]

Let Q_t be the Lagrange multiplier associated with constraint (3) if an investment opportunity arrives. It represents the shadow price of capital or Tobin's marginal Q. The following result characterizes firm j's optimization problem:

[13] Firms are subject to idiosyncratic investment efficiency shocks and are ex ante identical. Thus, b_t in ex ante firm value does not depend on firm-specific characteristics.

[14] According to the standard definition for exchange economies, a bubble is equal to the difference between the market value of an asset and the present value of the asset's *exogenously* given dividends. It is subtle to apply this definition to our model since dividends are *endogenously* generated through investment and production in our model. Bubbles can help firms make more investment and hence generate additional dividends. One criticism of the standard test for bubbles is that it is hard to separate bubbles from fundamentals in the data (see Gurkaynak (2008)). If one insists on the traditional definition of bubbles, one can call b_t the sunspot, self-fulfilling or speculative component without affecting our results. In a Bayesian DSGE model, Miao, Wang, and Xu (2012b) show that the fluctuations of this component can help explain the stock market volatility and the comovement between the stock market and real quantities.

Proposition 1 *Suppose* $Q_t > 1$ *and let* w_t *be given. Then the optimal investment level when an investment opportunity arrives is given by*

$$I_t^j = R_t K_t^j dt + \xi Q_t K_t^j + B_t, \tag{11}$$

where R_t *is given by (2) and*

$$B_t = e^{-rdt} b_{t+dt}, \tag{12}$$

$$Q_t = e^{-rdt} v_{t+dt}. \tag{13}$$

In addition,

$$v_t = R_t dt + (1 - \delta dt) Q_t + (Q_t - 1)(R_t dt + \xi Q_t) \pi dt, \tag{14}$$

$$b_t = B_t + (Q_t - 1) B_t \pi dt, \tag{15}$$

and the following transversality conditions hold:

$$\lim_{T\to\infty} e^{-rTdt} Q_T K_{T+dt}^j = 0, \quad \lim_{T\to\infty} e^{-rTdt} b_T = 0.$$

The intuition behind this proposition is as follows. When an investment opportunity arrives, an additional unit of investment costs the firm one unit of the consumption good, but generates an additional value of Q_t, where Q_t satisfies (13). This equation and equation (10) reveal that

$$Q_t = e^{-rdt} \frac{\partial V_{t+dt}(K_{t+dt})}{\partial K_{t+dt}}.$$

Thus, Q_t represents the marginal value of the firm following a unit increase in capital at time $t + dt$ in time-t dollars, i.e., Tobin's marginal Q. If $Q_t > 1$, the firm will make the maximal possible level of investment. If $Q_t = 1$, the investment level is indeterminate. If $Q_t < 1$, the firm will make the minimal possible level of investment. This investment choice is similar to Tobin's Q theory (Tobin (1969) and Hayashi (1982)). In what follows, we impose assumptions to ensure $Q_t > 1$ at least in the neighborhood of the steady state equilibrium. We thus obtain the investment rule given in (11). Substituting this rule and equation (10) into the Bellman equation (4) and matching coefficients, we obtain equations (14) and (15).

More specifically, we rewrite the firm's problem explicitly as:

$$\begin{aligned} v_t K_t^j + b_t &= \max_{I_t^j} R_t K_t^j dt - \pi I_t^j dt + \underbrace{e^{-rdt} v_{t+dt}}_{Q_t} \pi I_t^j dt \\ &\quad + \underbrace{e^{-rdt} v_{t+dt}}_{Q_t} (1 - \delta dt) K_t^j + \underbrace{e^{-rdt} b_{t+dt}}_{B_t}, \end{aligned}$$

subject to

$$I_t^j \leq R_t K_t^j dt + e^{-rdt} V_{t+dt}(\xi K_t^j) = R_t K_t^j dt + \underbrace{e^{-rdt} v_{t+dt}}_{Q_t} \xi K_t^j + \underbrace{e^{-rdt} b_{t+dt}}_{B_t}.$$

The expectation of a higher future firm value due to a bubble $b_{t+dt} > 0$ allows the borrowing constraint to be relaxed. Thus, bubbles are accompanied by a credit boom, leading the firm to make more investments by $B_t = e^{-rdt} b_{t+dt}$. This raises firm value by $\pi (Q_t - 1) B_t dt$ if $Q_t > 1$ and supports the inflated market value of assets in the sense that $b_t > 0$ must satisfy (15). This positive feedback loop mechanism generates a stock-price bubble $b_t > 0$ or $B_t > 0$ for all t.

Although our model features a constant-returns-to-scale technology, marginal Q is not equal to average Q in the presence of bubbles, because average Q is equal to

$$\frac{e^{-rdt} V_{t+dt} (K_{t+dt})}{K_{t+dt}} = Q_t + \frac{B_t}{K_{t+dt}}, \text{ for } B_t \neq 0.$$

Thus, the existence of stock price bubbles invalidates Hayashi's (1982) result. In the empirical investment literature, researchers typically use average Q to replace marginal Q under the constant-returns-to-scale assumption because marginal Q is not observable. Our analysis demonstrates that the existence of collateral constraints implies that stock prices may contain a bubble component that makes marginal Q not equal to average Q.

Next, we aggregate individual firm's decision rules and impose market-clearing conditions. We then characterize a competitive equilibrium by a system of nonlinear difference equations:

Proposition 2 *Suppose $Q_t > 1$. Then the equilibrium sequences (B_t, Q_t, K_t), for $t = 0$, dt, $2dt, \ldots$, satisfy the following system of nonlinear difference equations:*

$$B_t = e^{-rdt} B_{t+dt}[1 + \pi(Q_{t+dt} - 1)dt], \tag{16}$$

$$Q_t = e^{-rdt} \left[R_{t+dt} dt + (1 - \delta dt) Q_{t+dt} + (\xi Q_{t+dt} + R_{t+dt} dt)(Q_{t+dt} - 1)\pi dt\right], \tag{17}$$

$$K_{t+dt} = (1 - \delta dt) K_t + \pi (R_t K_t dt + \xi Q_t K_t + B_t) dt, \; K_0 \text{ given,} \tag{18}$$

and the transversality condition:

$$\lim_{T \to \infty} e^{-rTdt} Q_T K_{T+dt} = 0, \quad \lim_{T \to \infty} e^{-rTdt} B_T = 0,$$

where $R_t = \alpha K_t^{\alpha - 1}$.

When $dt = 1$, the above system reduces to the usual discrete-time characterization of equilibrium. However, this system is not convenient for analytically characterizing local dynamics. We may solve this system numerically by assigning parameter values. Instead of pursuing this route, we use analytical methods in the continuous-time limit as dt goes to zero. To compute the limit, we use the heuristic rule $dX_t = X_{t+dt} - X_t$ for any variable X_t. We also use the notation $\dot{X}_t = dX_t/dt$. We obtain the following:

Proposition 3 *Suppose $Q_t > 1$. Then in the continuous-time limit as $dt \to 0$, the equilibrium dynamics (B_t, Q_t, K_t) satisfy the following system of differential equations:*

$$\dot{B}_t = rB_t - B_t\pi(Q_t - 1), \tag{19}$$

$$\dot{Q}_t = (r + \delta) Q_t - R_t - \pi\xi Q_t(Q_t - 1), \tag{20}$$

$$\dot{K}_t = -\delta K_t + \pi(\xi Q_t K_t + B_t), \;\; K_0 \text{ given}, \tag{21}$$

and the transversality condition:

$$\lim_{T\to\infty} e^{-rT} Q_T K_T = 0, \quad \lim_{T\to\infty} e^{-rT} B_T = 0,$$

where $R_t = \alpha K_t^{\alpha-1}$. In addition, $Q_t = v_t$ and $B_t = b_t$ so that the market value of firm j is given by $V_t(K_t^j) = Q_t K_t^j + B_t$.

Equation (19) is an asset pricing equation for the bubble which will be explained in Section 5. Equation (20) is an asset pricing equation for capital. It says that the return on a unit of capital rQ_t is equal to the sum of the marginal product of capital R_t, the additional value generated from new investment $\pi(R_t + \xi Q_t)(Q_t - 1)$, and capital gains, minus the depreciation δQ_t. Equation (21) gives the law of motion for the aggregate capital stock.

After obtaining the solution for (B_t, Q_t, K_t), we can derive the equilibrium wage rate $w_t = (1-\alpha) K_t^{\alpha}$, marginal product of capital $R_t = \alpha K_t^{\alpha-1}$, aggregate output $Y_t = K_t^{\alpha}$, aggregate investment,

$$\pi I_t = \pi \left(\xi Q_t K_t + B_t\right), \tag{22}$$

and aggregate consumption $C_t = Y_t - \pi I_t$. We focus on two types of equilibrium.[15] The first type is bubbleless, for which $B_t = 0$ for all t. In this case, the market value of firm j is equal to its fundamental value in that $V_t(K_t^j) = Q_t K_t^j$. The second type is bubbly, for which $B_t > 0$

[15] We focus on the case where either all firms have bubbles in their stock prices or no firms have bubbles. It is possible to have another type of equilibrium in which only a fraction of firms have bubbles in their stock prices.

for some t. We assume that assets can be freely disposed of so that the bubbles B_t cannot be negative. In this case, firm value contains a bubble component in that $V_t(K_t^j) = Q_t K_t^j + B_t$ with $B_t > 0$. We next study these two types of equilibrium.

4 Bubbleless Equilibrium

In a bubbleless equilibrium, $B_t = 0$ for all t. Equation (19) becomes an identity. We only need to focus on (Q_t, K_t) determined by the differential equations (20) and (21) in which $B_t = 0$ for all t.

We first analyze the steady state. In the steady state, all aggregate variables are constant over time so that $\dot{Q}_t = \dot{K}_t = 0$. We use X to denote the steady state value of any variable X_t. By (20) and (21), we obtain the following steady-state equations:

$$\dot{Q} = 0 = (r + \delta) Q - R - \pi\xi Q(Q - 1), \tag{23}$$

$$\dot{K} = 0 = -\delta K + \pi(\xi Q K), \tag{24}$$

where $R = \alpha K^{\alpha-1}$. We use a variable with an asterisk to denote its value in the bubbleless equilibrium. Solving equations (23)-(24) yields:

Proposition 4 *(i) If*

$$\xi \geq \frac{\delta}{\pi}, \tag{25}$$

then there exists a unique bubbleless steady state equilibrium with $Q_t^* = Q_E \equiv 1$ *and* $K_t^* = K_E$, *where* K_E *is the efficient capital stock satisfying* $\alpha(K_E)^{\alpha-1} = r + \delta$.

(ii) If

$$0 < \xi < \frac{\delta}{\pi}, \tag{26}$$

then there exists a unique bubbleless steady-state equilibrium with

$$Q^* = \frac{\delta}{\pi}\frac{1}{\xi} > 1, \tag{27}$$

$$\alpha (K^*)^{\alpha-1} = \frac{r\delta}{\pi\xi} + \delta. \tag{28}$$

In addition, $K^* < K_E$.

Assumption (25) says that if firms pledge sufficient assets as collateral, then the collateral constraints will not bind in equilibrium. The competitive equilibrium allocation is the same as the efficient allocation. The efficient allocation is achieved by solving a social planner's problem in which the social planner maximizes the representative household's utility subject to the resource constraint only. Note that we assume that the social planner also faces stochastic investment opportunities, like firms in a competitive equilibrium. Thus, one may view our definition of the efficient allocation as the constrained efficient allocation. Unlike firms in a competitive equilibrium, the social planner is not subject to collateral constraints.

Assumption (26) says that if firms do not pledge sufficient assets as collateral, then the collateral constraints will be sufficiently tight so that firms are credit constrained in the neighborhood of the steady-state equilibrium in which $Q^* > 1$. We can then apply Proposition 3 in this neighborhood. Proposition 4 also shows that the steady-state capital stock for the bubbleless competitive equilibrium is less than the efficient steady-state capital stock. This reflects the fact that not enough resources are transferred from savers to investors due to the collateral constraints.

Note that for (26) to hold, the arrival rate π of the investment opportunity must be sufficiently small, holding everything else constant. The intuition is that if π is too high, then too many firms will have investment opportunities so that the accumulated aggregate capital stock will be sufficiently large, thereby lowering the capital price Q to the efficient level as shown in part (i) of Proposition 4. In this case, firms can accumulate sufficient internal funds and do not need external financing. Thus, the collateral constraints will not bind and the economy will reach the first best. Condition (26) requires that technological constraints at the firm level be sufficiently tight.

Now, we study the stability of the bubbleless steady state and the dynamics of the equilibrium system. We use the phase diagram in Figure 2 to describe the two-dimensional dynamic system for (Q_t, K_t). The $\dot{Q} = 0$ locus and the $\dot{K} = 0$ locus intersect once at the bubbleless steady state by Proposition 4. It is straightforward to show that the $\dot{K} = 0$ locus is vertical (24). To the right of this locus, $\dot{K} > 0$, and to the left of this locus $\dot{K} < 0$. The intuition is that to right of the $\dot{K} = 0$ locus, Q is large for a given capital stock. Firms prefer to invest more so that capital rises.

Turning to the $\dot{Q} = 0$ locus, one can verify that on the $\dot{Q} = 0$ locus, $dK/dQ|_{Q\to 1} < 0$ and $dK/dQ|_{Q\to\infty} > 0$. But for general values of $Q > 1$, we cannot determine the sign of dK/dQ. We can show that, above the $\dot{Q} = 0$ locus, $\dot{Q} > 0$, and below the $\dot{Q} = 0$ locus, $\dot{Q} < 0$. The intuition is that above the $\dot{Q} = 0$ locus, the capital stock is high for a fixed Q. This implies that the marginal product of capital is low. To maintain the return on capital in the asset pricing equation (20), the price of capital Q must rise. In summary, there are two cases depending on

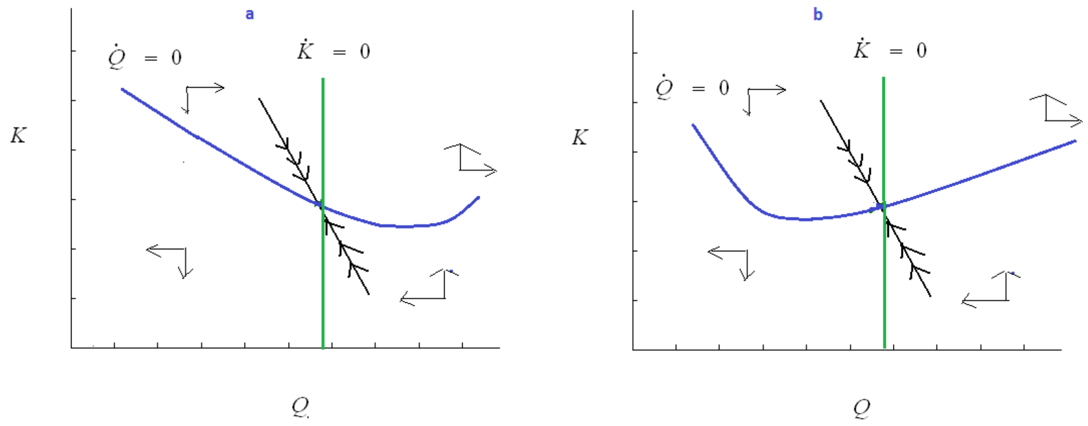

Figure 2: Phase diagram for the dynamics of the bubbleless equilibrium. The left (right) panel illustrates the case in which the $\dot{Q} = 0$ locus crosses the $\dot{K} = 0$ from above (below).

whether the $\dot{Q} = 0$ locus crosses the $\dot{K} = 0$ locus from below or from above, as illustrated in Figure 2. For both cases, there is a unique saddle path such that for any given initial value K_0, when Q_0 is on the saddle path, the economy approaches the long-run steady state (Q^*, K^*).[16]

5 Bubbly Equilibrium

In this section, we study the bubbly equilibrium in which $B_t > 0$ for all t. We shall analyze the dynamic system for (B_t, Q_t, K_t) given in (19)-(21). Before we conduct a formal analysis later, we first explain why bubbles can exist in our model. The key lies in understanding equation (19), rewritten here as

$$\frac{\dot{B}_t}{B_t} + \pi(Q_t - 1) = r, \text{ for } B_t \neq 0. \tag{29}$$

The first term on the left-hand side is the rate of capital gains of bubbles. The second term represents a "collateral yield," as we will explain below. Thus, equation (19) or (29) reflects a no-arbitrage relation in that the rate of return on bubbles must be equal to the interest rate. A similar relation also appears in the literature on rational bubbles, e.g., Blanchard and Watson (1982), Tirole (1985), and Weil (1987). This literature typically studies bubbles on zero-payoff assets or unproductive assets with exogenously given payoffs. In this case, the

[16] A formal proof is available upon request.

second term on the left-hand side of (29) vanishes and bubbles grow at the rate r of interest on the household bonds. If we adopt collateral constraint (9) as in Kiyotaki and Moore (1997), then we can also show that bubbles grow at the rate of interest r. In an infinite-horizon economy, the transversality condition rules out these bubbles. In an overlapping generations economy, for bubbles to exist, the interest rate must be less than the growth rate of the economy in the bubbleless equilibrium. This means that the bubbleless equilibrium must be dynamically inefficient (see Tirole (1985)).

In line with this literature, equation (16) reveals that the existence of a bubble today ($B_t > 0$) in our model depends on people's expectations that it will be valuable in the future (i.e., $B_{t+dt} > 0$). If the bubble is expected to burst in the future ($B_{t+dt} = 0$), then it has no value today ($B_t = 0$). Unlike this literature, bubbles in our model are attached to productive real assets and also influence their fundamentals (or dividends). Specifically, each unit of a bubble raises the collateral value by one unit and hence allows the firm to borrow an additional unit. The firm then makes one more unit of investment when an investment opportunity arrives. This unit of investment raises firm value by $Q_t > 1$. Subtracting one unit of costs, we then deduce that the second term on the left-hand side of (29) represents the net increase in firm value for each unit of a bubble. This is why we call this term the collateral yield. The collateral yield causes the growth rate of bubbles to be lower than the interest rate. Thus, the transversality conditions cannot rule out bubbles in our model. We can also show that the bubbleless equilibrium is dynamically efficient in our model. Specifically, the golden rule capital stock is given by $K_{GR} = (\delta/\alpha)^{\frac{1}{\alpha-1}}$. One can verify that $K^* < K_{GR}$. Thus, one cannot use the condition for the overlapping generations economies in Tirole (1985) to ensure the existence of bubbles. Next, we will give new conditions to ensure the existence of bubbles in our model.

5.1 Steady State

We first study the existence of a bubbly steady state in which $B > 0$. We use a variable with a subscript b to denote this variable's bubbly steady state value. By Proposition 3, (B, Q_b, K_b) satisfies equations (23),

$$0 = rB - B\pi(Q - 1), \text{ and} \tag{30}$$

$$0 = -\delta K + [\xi QK + B]\pi, \tag{31}$$

where $R = \alpha K^{\alpha-1}$. Using these equations, we can derive:

Proposition 5 *There exists a bubbly steady state satisfying*

$$\frac{B}{K_b} = \frac{\delta}{\pi} - \xi(\frac{r}{\pi} + 1) > 0, \tag{32}$$

$$Q_b = \frac{r}{\pi} + 1 > 1, \tag{33}$$

$$\alpha (K_b)^{\alpha - 1} = [(1 - \xi)r + \delta](\frac{r}{\pi} + 1), \tag{34}$$

if and only if the following condition holds:

$$0 < \xi < \frac{\delta}{r + \pi}. \tag{35}$$

In addition, (i) $Q_b < Q^*$, (ii) $K_{GR} > K_E > K_b > K^*$, and (iii) the bubble-asset ratio B/K_b decreases with ξ.

From equations (23), (30) and (31), we can immediately derive (32)-(34). We can then immediately see that condition (35) is equivalent to $B/K_b > 0$. This condition reveals that bubbles occur when ξ is sufficiently small, *ceteris paribus.*[17] The intuition is as follows. When the degree of pledgeability is sufficiently low, the credit constraint is too tight and a bubble can help relax this constraint. This allows firms to borrow more and invest more. If the collateral constraint is not tight enough, firms can borrow sufficient funds to finance investment. In this case, a bubble serves no function.

Note that condition (35) implies condition (26). Thus, if condition (35) holds, then there exist two steady state equilibria: one bubbleless and the other bubbly. The bubbleless steady state is analyzed in Proposition 4. Propositions 5 and 4 reveal that the steady-state capital price is lower in the bubbly equilibrium than in the bubbleless equilibrium, i.e., $Q_b < Q^*$. The intuition is as follows. In a bubbleless or a bubbly steady state, the investment rate must be equal to the rate of capital depreciation such that the capital stock is constant over time (see equations (24) and (31)). Bubbles help relax collateral constraints and induce firms to make more investment than in the case without bubbles. To maintain the same steady-state investment rate, the capital price in the bubbly steady state must be lower than that in the bubbleless steady state.

Do bubbles crowd out capital in the steady state? In Tirole's (1985) overlapping generations model, households may use part of their savings to buy bubble assets instead of accumulating

[17] This result depends on our modeling of idiosyncratic investment opportunity shocks. In Section 7.2, we show that bubbles in stock prices can exist even for $\xi = 1$ when firms face idiosyncratic investment efficiency shocks with a continuous distribution.

capital. Thus, bubbles crowd out capital in the steady state. In our model, bubbles are on productive assets. If the capital price were the same for both bubbly and bubbleless steady states, then bubbles would induce firms to invest more and hence to accumulate more capital stock. However, there is a general equilibrium price feedback effect as discussed earlier. The lower capital price in the bubbly steady state discourages firms to accumulate more capital stock. The net effect is that bubbles lead to higher capital accumulation, unlike Tirole's (1985) result. Note that bubbles do not lead to efficient allocation. The capital stock in the bubbly steady state is still lower than that in the efficient allocation.

How does the pledgeability parameter ξ affect the size of bubbles? Proposition 5 shows that a smaller ξ leads to a larger bubble relative to capital in the steady state. This is intuitive. If firms can only pledge a smaller amount of assets, they will face a tighter collateral constraint so that a larger bubble is needed to relax this constraint.

5.2 Dynamics

Now, we study the stability of the two steady states and the local dynamics around them. Since the equilibrium system (19)-(21) is three dimensional, we cannot use the phase diagram to analyze its stability. We thus consider a linearized system and obtain the following:

Proposition 6 *Suppose condition (35) holds. Then both the bubbly steady state* (B, Q_b, K_b) *and the bubbleless steady state* $(0, Q^*, K^*)$ *are local saddle points for the nonlinear system (19)-(21).*

More formally, in Appendix A, we prove that for the nonlinear system (19)-(21), there is a neighborhood $\mathcal{N} \subset \mathbb{R}^3_+$ of the bubbly steady state (B, Q_b, K_b) and a continuously differentiable function $\phi : \mathcal{N} \to \mathbb{R}^2$ such that given any K_0 there exists a unique solution (B_0, Q_0) to the equation $\phi(B_0, Q_0, K_0) = 0$ with $(B_0, Q_0, K_0) \in \mathcal{N}$, and (B_t, Q_t, K_t) converges to (B, Q_b, K_b) starting at (B_0, Q_0, K_0) as t approaches infinity. The set of points (B, Q, K) satisfying the equation $\phi(B, Q, K) = 0$ is a one-dimensional stable manifold of the system. If the initial value (B_0, Q_0, K_0) is on the stable manifold, then the solution to the nonlinear system (19)-(21) is also on the stable manifold and converges to (B, Q_b, K_b) as t approaches infinity.

Although the bubbleless steady state $(0, Q^*, K^*)$ is also a local saddle point, the local dynamics around this steady state are different. In Appendix A, we prove that the stable manifold for the bubbleless steady state is two dimensional. Formally, there is a neighborhood $\mathcal{N}^* \subset \mathbb{R}^3_+$ of $(0, Q^*, K^*)$ and a continuously differentiable function $\phi^* : \mathcal{N}^* \to \mathbb{R}$ such that given any (B_0, K_0) there exists a unique solution Q_0 to the equation $\phi^* (B_0, Q_0, K_0) = 0$ with $(B_0, Q_0, K_0) \in \mathcal{N}$, and (B_t, Q_t, K_t) converges to $(0, Q^*, K^*)$ starting at (B_0, Q_0, K_0) as t approaches infinity. Intuitively, along the two-dimensional stable manifold, the bubbly equilibrium is asymptotically bubbleless in that bubbles will burst eventually.

6 Alternative Assets

So far, we have assumed that there are two types of assets for households to trade without frictions: risk-free household bonds and stocks. But firms cannot trade household bonds and can only take intratemporal loans subject to credit constraints. In this section, we introduce other types of assets and study the conditions under which a bubble can exist in the presence of different types of assets. We first consider intertemporal corporate bonds in Section 6.1. We then consider assets with rents and assets with a zero market fundamental in Section 6.2. The latter assets can be thought of as pieces of paper, or pure bubbles. It is important to note that no arbitrage implies that all traded assets must earn the same rate of return r in equilibrium since there is no aggregate uncertainty in the economy. We shall show that both households and firms must face market frictions for a bubble to exist in an infinite-horizon economy. A similar point is made by Kocherlakota (1992, 2009) and Santos and Woodford (1997) for pure-exchange economies.

6.1 Intertemporal Borrowing and Savings

Suppose that there is no intratemporal loan. But firms can borrow or save by selling or buying intertemporal bonds, respectively. Firms can use these bonds to finance investments subject to credit constraints. These bonds are in zero net supply. Households can also trade these bonds, but are subject to borrowing or short-sales constraints. We may reinterpret the corporate bonds as a bank account. Both households and firms can borrow from or save in it. We will show below that, without household borrowing constraints, no arbitrage implies that the economy would achieve the efficient equilibrium and no bubble could exist.

We will derive an equilibrium in which firms with investment opportunities choose to borrow, firms without investment opportunities choose to save, and households are borrowing constrained. We will also show that the equilibrium interest rate on the corporate bonds is less than r, because these bonds provide liquidity to the firms and demand a liquidity premium. We will still analyze a discrete-time approximation and then study its continuous-time limit.

Let the interest rate on the corporate bonds between time t and $t+dt$ be $r_{ft}dt$. Let L_t^h denote households' bond holdings. The short-sales constraint for the households is given by $L_t^h \geq 0$ for all t. Let L_t^j denote firm j's debt level. When $L_t^j < 0$, L_t^j means savings. Let $V_t\left(K_t^j, L_t^j\right)$ denote the ex ante equity value of a typical firm j when its capital stock and debt at time t are K_t^j and L_t^j, respectively, prior to the realization of the investment opportunity shock. We suppress the aggregate state variables in the argument. Then V_t satisfies the following Bellman equation:

$$\begin{aligned} V_t\left(K_t^j, L_t^j\right) &= \pi dt \max_{L_{t+dt}^j, I_t^j \geq 0} \left\{D_t^j + e^{-rdt} V_{t+dt}(K_{t+dt}^j, L_{t+dt}^j)\right\} \\ &\quad + (1-\pi dt) \max_{L_{t+dt}^j} \left\{D_t^j dt + e^{-rdt} V_{t+dt}((1-\delta dt)K_t^j, L_{t+dt}^j)\right\}, \end{aligned} \tag{36}$$

where the first max operator is subject to the constraints

$$D_t^j + I_t^j + L_t^j = R_t K_t^j dt + e^{-r_{ft}dt} L_{t+dt}^j, \tag{37}$$

$$K_{t+dt}^j = K_t^j(1-\delta dt) + I_t^j, \tag{38}$$

$$V_{t+dt}(K_{t+dt}^j, L_{t+dt}^j) \geq V_{t+dt}\left(K_{t+dt}^j, 0\right) - V_{t+dt}(\xi K_{t+dt}^j, 0), \tag{39}$$

and the second max operator is subject to the constraint[18]

$$D_t^j dt + L_t^j = R_t K_t^j dt + e^{-r_{ft}dt} L_{t+dt}^j. \tag{40}$$

We also impose the nonnegative dividend constraint $D_t^j \geq 0$.

The interpretation of the Bellman equation (36) is as follows. When an investment opportunity arrives at time $t + dt$ with probability πdt, firm j borrows $e^{-r_{ft}dt} L_{t+dt}^j \geq 0$ and makes investments I_t^j. The capital stock becomes K_{t+dt}^j as in (38). When no investment opportunity arrives with probability $1 - \pi dt$, the firm chooses to save $e^{-r_{ft}dt} L_{t+dt}^j \leq 0$. Debt is subject to the collateral constraint (39). Using other types of credit constraint as in Section 7 will not change our key insights. The interpretation of (39) is similar to that of (6). Suppose that at time t, firm j pledges a fraction ξ of its capital K_{t+dt}^j as collateral. It may default on debt L_{t+dt}^j at the beginning of period $t + dt$ before the realization of the investment opportunity shock. If it does not default, it obtains continuation value $V_{t+dt}\left(K_{t+dt}^j, L_{t+dt}^j\right)$. If it defaults, debt is renegotiated and the repayment L_{t+dt}^j is relieved. The lender can seize the collateralized assets ξK_{t+dt}^j and keep the firm running with these assets by reorganizing the firm. Thus, the threat

[18] In (40) for the case without an investment opportunity, the dividend payment is a flow.

value to the lender is $V_{t+dt}(\xi K_{t+dt}^j, 0)$. Assume that firms have full bargaining power. Then the expression on the right-hand side of (39) is the value to the firm if it chooses to default. Thus, the constraint (39) ensures that firm j does not have an incentive to default.

Equations (37) and (40) are the flow-of-funds constraints. Given the nonnegative dividend constraint, we obtain

$$0 \leq I_t^j \leq R_t K_t^j dt + e^{-r_{ft}dt} L_{t+dt}^j - L_t^j, \tag{41}$$

where we have assumed that investment cannot be negative, as in Section 2. It turns out that this constraint will not bind in equilibrium.

In Appendix A, we show that equity value in the continuous-time limit takes the following form:

$$V_t\left(K_t^j, L_t^j\right) = Q_t K_t^j - L_t^j + B_t,$$

and the credit constraint (39) becomes

$$L_t^j \leq \xi Q_t \left(K_t^j + I_t^j\right) + B_t,$$

where Q_t is the shadow price price of capital, B_t is the speculative component of equity value, and L_t^j is the amount of loans taken when an investment opportunity arrives.[19] As in the baseline model, the bubble B_t relaxes the credit constraint. The following proposition characterizes the equilibrium system:

Proposition 7 *Consider the continuous-time limit. Suppose that $1 < Q_t < 1/\xi$. Then the equilibrium system for (K_t, Q_t, B_t, r_{ft}) is given by*

$$\dot{K}_t = -\delta K_t + \pi \frac{\xi Q_t K_t + B_t}{1 - \xi Q_t}, \tag{42}$$

$$\dot{Q}_t = Q_t(r + \delta) - R_t - \pi \frac{\xi Q_t (Q_t - 1)}{1 - \xi Q_t}, \tag{43}$$

$$r B_t = \dot{B}_t + \pi \frac{Q_t - 1}{1 - \xi Q_t} B_t, \tag{44}$$

$$r_{ft} = r - \pi \frac{Q_t - 1}{1 - \xi Q_t} < r, \tag{45}$$

where $R_t = \alpha K_t^{\alpha - 1}$, and the usual transversality conditions hold.

[19] The credit constraint (39) is equivalent to

$$e^{-rdt} V_{t+dt}(K_{t+dt}^j, L_{t+dt}^j) \geq (1 - \xi)\, Q_t K_{t+dt}^j.$$

This constraint admits the following alternative interpretation. After default, the firm's capital is liquidated at the value $Q_t K_{t+dt}^j$. The lender gets a fraction ξ of the value and the firm gets the remaining value. The above inequality ensures that the firm has no incentive to default.

Equation (44) is an asset pricing equation for the bubble if it exists (i.e., $B_t > 0$). As in Section 2, the rate of return r on the bubble is equal to the capital gains (or the growth rate of the bubble) plus a collateral yield, $\pi\left(Q_t - 1\right) / \left(1 - \xi Q_t\right)$. Equation (45) shows that the rate of return r on the corporate bonds is equal to the interest rate r_{ft} plus a liquidity premium, $\pi\left(Q_t - 1\right) / \left(1 - \xi Q_t\right)$. The liquidity premium exists because the corporate bonds can provide liquidity to firms and help them finance investment. Because both the bubble and the corporate bonds can help firms finance investment to the same extent, the collateral yield and the liquidity premium are equal. Since we have assumed that firms cannot trade household bonds, households bonds do not carry a liquidity premium. Thus, the interest rate r on the household bonds is higher than the interest rate r_{ft} on the corporate bonds. The difference is the liquidity premium.

Since $r > r_{ft}$, households prefer to borrow by selling corporate bonds until their short-sales constraints bind. Without a short-sale constraint, households keep selling these bonds until $r_{ft} = r$. In this case, the liquidity premium is zero so that $Q_t = 1$ and the economy reaches the efficient equilibrium and no bubble can exist.

Next, we characterize the steady state.

Proposition 8 *(i) If*

$$0 < \xi < \frac{\delta}{\delta + \pi}, \tag{46}$$

then there exists a bubbleless steady state $\left(K^*, Q^*, r_f^*\right)$ *such that*

$$Q^* = \frac{\delta}{\pi + \delta} \frac{1}{\xi} \in \left(1, 1/\xi\right), \quad r_f^* = r + \pi + \delta - \delta/\xi,$$

$$\alpha \left(K^*\right)^{\alpha - 1} = R^* = \frac{\delta r}{\pi + \delta} \frac{1}{\xi} + \delta.$$

(ii) If

$$0 < \xi < \frac{\delta}{\delta + \pi + r}, \tag{47}$$

then there exists a bubbly steady state $\left(K_b, Q_b, B, r_f\right)$ *given by*

$$Q_b = \frac{r + \pi}{\xi r + \pi} \in \left(1, 1/\xi\right), \quad r_f = 0,$$

$$\alpha K_b^{\alpha - 1} = R_b = \left(r\left(1 - \xi\right) + \delta\right) \frac{r + \pi}{\xi r + \pi},$$

$$\frac{B}{K_b} = \frac{\delta - \xi\left(\delta + r + \pi\right)}{\xi r + \pi}.$$

Conditions (46) and (47) ensure that the assumption in Proposition 7 is satisfied in a neighborhood of the steady state. Proposition 8 shows that in a bubbly steady state, the interest rate on corporate bonds must be equal to zero. This is because both corporate bonds and bubbles can be used to finance investment. But the bubble is intrinsically useless. If the interest rate on corporate bonds is positive, then corporate bonds clearly dominate the bubble and a bubble cannot exist. This result is related to the rate-of-return-dominance puzzle in monetary economics. We may reinterpret the corporate bonds as a bank account, and both households and firms can borrow from and save in it. Proposition 8 shows that the equilibrium interest rate on the bank account must be equal to zero for a bubble and the bank account to coexist. Kocherlakota (2009) derives a similar result.

One may introduce economic growth to generate the coexistence of a bubble and a positive-interest-rate corporate bond as in Tirole (1985). Formally, let the production function be

$$Y_t^j = (K_t^j)^\alpha (A_t N_t^j)^{1-\alpha},$$

where $A_t = e^{gt}$ $(g > 0)$ represents technical progress. We can then show that aggregate capital and the firm bubble grow at the same rate g. Proposition 7 still characterizes the equilibrium system except that now $R_t = \alpha \left(K_t/A_t\right)^{\alpha-1}$. It follows from (44) and (45) that the steady-state interest rate on the corporate bond is equal to $g > 0$.

6.2 Assets with or without Rents

We now follow Tirole (1985) and introduce an asset that brings a real rent (or dividend) to the baseline model in Section 2. For simplicity, suppose that the rent is given by a constant $X \geq 0$ each time. If $X = 0$, then this asset has no market fundamental and is called a pure bubble (e.g., tulip). If $X > 0$, we call it a "tree." Firms can invest in the asset and sell it to finance its investment when an investment opportunity arrives. We normalize the total supply of the asset to unity. We follow Kocherlakota (1992, 2009) and assume that both households and firms face short-sales constraints. We will show that households will sell the asset until the short-sales constraints bind.

Since we have shown that intertemporal borrowing is not essential for the existence of a bubble, we shall focus on the case of intratemporal loans for simplicity. Let $V_t(K_t^j, M_t^j)$ denote the ex ante market value of a typical firm j when its capital stock and asset holdings at time t are K_t^j and M_t^j, respectively. Let P_t denote the market price of the asset. Then V_t satisfies the following Bellman equation:

$$\begin{aligned} V_t\left(K_t^j, M_t^j\right) &= \pi dt \max_{M_{t+dt}^j \geq 0, I_t^j \geq 0} \left\{D_t^j - L_t^j + e^{-rdt} V_{t+dt}(K_{t+dt}^j, M_{t+dt}^j)\right\} \\ &\quad + (1-\pi dt) \max_{M_{t+dt}^j \geq 0} \left\{D_t^j dt + e^{-rdt} V_{t+dt}((1-\delta dt)K_t^j, M_{t+dt}^j)\right\}, \end{aligned} \tag{48}$$

where the first max operator is subject to the constraint (38),

$$D_t^j + I_t^j + P_t M_{t+dt}^j = R_t K_t^j dt + (P_t + X dt) M_t^j + L_t^j, \ D_t^j \geq 0, \text{ and}$$

$$L_t^j \leq e^{-rdt} V_{t+dt}\left(\xi K_t^j, 0\right), \tag{49}$$

and the second max operator is subject to the constraint

$$D_t^j dt + P_t M_{t+dt}^j = R_t K_t^j dt + (P_t + X dt) M_t^j, \ D_t^j \geq 0.$$

The interpretations of the Bellman equation and the constraints are similar to those in Sections 2 and 6.1. In particular, equation (49) gives the collateral constraints. Since firm j can sell the asset directly to finance investment when an investment opportunity arrives, it uses ξK_t^j as collateral only. Alternatively, the firm can use the asset as collateral to borrow and hence the flow-of-funds constraint and the collateral constraint become

$$D_t^j + I_t^j + P_t M_{t+dt}^j = R_t K_t^j dt + L_t^j, \tag{50}$$

$$L_t^j \leq e^{-rdt} V_{t+dt}\left(\xi K_t^j, M_t^j\right) + M_t^j X_t dt. \tag{51}$$

The right-hand side of (51) gives the threat value or the recovery value to the lender if the firm defaults. In Appendix A, we show that the above two ways of financing deliver an identical equilibrium outcome.

In Appendix A, we also show that V_t takes the following form:

$$V_t\left(K_t^j, M_t^j\right) = Q_t K_t^j + P_t M_t^j + B_t,$$

where Q_t, P_t, and B_t are characterized in the following proposition:

Proposition 9 *Suppose $Q_t > 1$. Then the equilibrium system for (K_t, Q_t, B_t, P_t) is given by*

$$\dot{K}_t = -\delta K_t + \pi(Q_t \xi K_t + P_t + B_t), \tag{52}$$

$$\dot{Q}_t = (r+\delta) Q_t - R_t - \pi(Q_t - 1) Q_t \xi, \tag{53}$$

$$\dot{B}_t = r B_t - \pi(Q_t - 1) B_t, \tag{54}$$

$$\dot{P}_t = r P_t - X - \pi(Q_t - 1) P_t, \tag{55}$$

where $R_t = \alpha K_t^{\alpha - 1}$, and the usual transversality conditions hold.

The interpretations of equations (52)-(54) are similar to those in Section 2. As in Section 2, we interpret $\pi(Q_t - 1)$ as the collateral yield. Equation (55) is an asset pricing equation. We rewrite it as

$$r = \frac{\dot{P}_t}{P_t} + \frac{X}{P_t} + \pi(Q_t - 1) \text{ for } P_t > 0,$$

which implies that the rate of return r on the asset consists of three components: (i) capital gains, (ii) dividend yield, and (iii) collateral yield $\pi(Q_t - 1)$. The asset can generate a collateral yield when $Q_t > 1$. In this case, the firm is credit constrained and has to either sell the asset or use it as collateral to finance investment.

Since $rP > \dot{P}_t + X$, households have no incentive to hold the asset and want to sell it until their short-sales constraints bind. If there were no household short-sales constraint, then no arbitrage would force the collateral yield to vanish so that $Q_t = 1$. In this case, there would be no investment friction and hence the economy reaches the efficient equilibrium in which no bubble can exist.

In the special case of $X = 0$, the asset is intrinsically useless and becomes a pure bubble with a zero market fundamental. Equation (55) reduces to

$$\dot{P}_t = rP_t - \pi(Q_t - 1)P_t.$$

Comparing this equation with (54) reveals that the bubble value P_t and the component B_t in firm value follow the same asset pricing equation. By (52), we can determine the total value $P_t + B_t$ only, but not B_t or P_t separately. That is, the bubble asset and the component B_t in firm value are perfect substitutes if the latter can be traded in the market. This justifies our interpretation of B_t as a bubble. In this case, there are multiple equilibria which can be characterized as in Propositions 4 and 5. The bubbly equilibrium determines the total size of the bubble, but the decomposition of the total bubble is indeterminate. The equilibrium real allocation is independent of the decomposition. This result is analogous to that discussed in Section 5 of Tirole (1985).

In the case of $X > 0$, a bubbly equilibrium cannot exist because the tree with dividends dominates the bubble. To see this, suppose that a bubble exists in the steady state. Then (54) and (55) imply the steady-state relation

$$0 = r - \pi(Q - 1) = X/P > 0,$$

which is a contradiction. The intuition is that the tree with positive dividends plays the same role as a firm bubble in that both can be used to finance investment. The tree dominates the bubble since it delivers positive dividends, but the bubble has a zero market fundamental.

For a bubble and a tree to coexist, we may introduce economic growth to the model by allowing technical progress as in Section 6.1. In this case, Proposition 9 still applies except that $R_t = \alpha \left(K_t/A_t\right)^{\alpha-1}$. We then obtain

$$\begin{aligned} \dot{b}_t &= (r-g)b_t - \pi(Q_t - 1)b_t, \\ \dot{p}_t &= (r-g)p_t - \pi(Q_t - 1)p_t - X/e^{gt}, \end{aligned}$$

where $b_t = B_t/A_t$ and $p_t = P_t/A_t$. Since the detrended dividend X/e^{gt} vanishes in the long run, the firm bubble and the tree can coexist in the steady state. More generally, if dividends grow at a lower rate than the economy does, then dividends normalized by the trend disappears in the long-run. In this case, the tree can still coexist with the firm bubble.

7 Alternative Credit Constraints

In this section, we consider some other types of credit constraints in the baseline model of Section 2 with intratemporal loans. We shall show that our key insight that bubbles can help relax credit constraints still holds. We shall demonstrate that bubbles can exist for a variety of endogenous credit constraints with limited commitment.

7.1 Self-Enforcing Constraints

Consider a different type of credit constraint which is popular in the self-enforcing debt literature (see, e.g., Bulow and Rogoff (1989), Kehoe and Levine (1993), Alvarez and Jermann (2000), Albuquerque and Hopenhayn (2004), Kocherlakota (2008), and Hellwig and Lorenzoni (2009)).[20] There is no collateral. Suppose that the only penalty on the firm for defaulting is that it is excluded from the financial market forever. In this case, all the future investment spending is financed by internal funds only. Denote by $V_t^a(K_t^j)$ the autarky value of firm j that cannot access the financial market. $V_t^a(K_t^j)$ solves an approximated discrete-time Bellman equation (4), subject to the investment constraint, $0 \leq I_t^j \leq R_t K_t^j dt$. This is a standard dynamic programming problem and a bubble cannot exist after default by the transversality condition.

Let us now turn to firm j's decision problem before it defaults. Firm value $V_t(K_t^j)$ solves the Bellman equation (4) subject to the investment constraint (5) and the following credit

[20] Kocherlakota (2008) and Hellwig and Lorenzoni (2009) show that a bubble can exist with self-enforcing debt constraints in a pure exchange economy.

constraint analogous to (7),

$$-L_t^j + e^{-rdt} V_{t+dt}((1-\delta dt) K_t^j + I_t^j) \geq e^{-rdt} V_{t+dt}^a((1-\delta dt) K_t^j + I_t^j). \tag{56}$$

This credit constraint is an incentive constraint which can be interpreted as follows. When an investment opportunity arrives, firm j takes debt L_t^j and makes investment I_t^j. If it repays the debt, its continuation value is given by the expression on the left-hand side of (56). If it defaults on the debt, it is excluded from the financial market forever and its continuation value is given by the expression on the right-hand side of (56). Inequality (56) ensures that the firm has no incentive to default.

We rewrite (56) as

$$L_t^j \leq e^{-rdt} V_{t+dt}((1-\delta dt) K_t^j + I_t^j) - e^{-rdt} V_{t+dt}^a((1-\delta dt) K_t^j + I_t^j). \tag{57}$$

As in the baseline model, if both the lender and the firm believe that the future firm value V_{t+dt} contains a bubble, then the bubble can relax the credit constraint and allow the firm to borrow more. Thus, the firm can finance more investment and raise its market value, justifying the optimistic beliefs about the bubble.

In Appendix B.1, we analyze equilibrium in the continuous-time limit. We show that if $Q_t > 1$, the endogenous debt limit is equal to B_t, where B_t satisfies the asset pricing equation (19). This result is similar to Theorem 1 in Hellwig and Lorenzoni (2009), which states that the endogenous debt limit satisfies an exact roll-over condition in an endowment economy. We also show that Q_t and K_t satisfy differential equations (20) and (21) for $\xi = 0$. Note that condition (35) applies to $\xi = 0$. Thus, Proposition 5 is still valid here and both bubbleless and bubbly equilibria can exist. The bubbleless equilibrium is autarky. The bubbly equilibrium with self-enforcing debt constraints is a special case of that in the baseline model with collateral constraints in which an empty firm with zero assets is pledged as collateral or the bubble is effectively pledged as collateral. This result is reminiscent to the modeling of credit constraints in Martin and Ventura (2011, 2012).

7.2 Firm Value as Collateral

In the baseline model, we have assumed that any firm j can pledge a fraction of its assets ξK_t^j as collateral. The collateral value is equal to the market value $V_t\left(\xi K_t^j\right)$ of the firm with these assets. We now assume that the firm can pledge a fraction ξ of its market value as collateral directly. In this case, the credit constraint for firm j in continuous time becomes[21]

$$L_t^j \leq \xi V_t\left(K_t^j\right). \tag{58}$$

This collateral constraint and (8) in the baseline model are identical when $\xi = 1$.

We also assume that when investment opportunities arrive at date t at Poisson arrival rate π, firm j faces an idiosyncratic investment efficiency shock ε_t^j, which is independently drawn from a distribution Φ on $[\varepsilon_{\min}, \varepsilon_{\max}]$. After observing the investment efficiency shock, firm j chooses an investment level I_t^j, which is a function of ε_t^j. The law of motion for capital is given by

$$K_{t+dt}^j = \begin{cases} (1-\delta dt)\, K_t^j + \varepsilon_t^j I_t^j & \text{with probability } \pi dt \\ (1-\delta dt)\, K_t^j & \text{with probability } 1-\pi dt \end{cases}. \tag{59}$$

In the baseline model, we have shown that whenever an investment opportunity arrives, a firm invests at its maximal level (see Proposition 1). In Appendix B.2, we show that facing idiosyncratic investment efficiency shocks, a firm invests only when the investment efficiency shocks exceed a cutoff value. The presence of a bubble can change this cutoff value. Thus, bubbles can affect both the size of investment (intensive margin) and the number of investing firms (extensive margin).

In Appendix B.2, we show that the equilibrium system is given by

$$rB_t = \dot{B}_t + B_t \pi \xi \int_{\varepsilon_t^*}^{\varepsilon_{\max}} (Q_t \varepsilon - 1) d\Phi(\varepsilon), \tag{60}$$

$$rQ_t = -\delta Q_t + R_t + \dot{Q}_t + \pi \xi Q_t \int_{\varepsilon_t^*}^{\varepsilon_{\max}} (Q_t \varepsilon - 1) d\Phi(\varepsilon), \tag{61}$$

$$\dot{K}_t = -\delta K_t + \pi \xi \left(Q_t K_t + B_t\right) \int_{\varepsilon_t^*}^{\varepsilon_{\max}} \varepsilon d\Phi(\varepsilon), \tag{62}$$

where $R_t = \alpha K_t^{\alpha-1}$ and $\varepsilon_t^* = 1/Q_t$ is the cutoff value for investment. The usual transversality conditions must also hold.

In a bubbleless equilibrium in which $B_t = 0$, equation (60) is automatically satisfied. The question is whether there exists a bubbly equilibrium in which $B_t > 0$. As in the baseline model, the presence of a bubble relaxes credit constraints and allows firms to make more investment, thereby generating higher firm value. As shown in equation (60), a one-dollar bubble can generate additional profits, $\pi\xi \int_{\varepsilon_t^*}^{\varepsilon_{\max}} (Q_t \varepsilon - 1) d\Phi(\varepsilon)$, for the firm. Unlike equation (19) in the

[21] Note that we have abused the notation V_t since firm value may take a different functional form in this case than in the baseline model.

baseline model, the bubble here has an extensive margin effect on investment in the sense that firms invest if and only if their investment efficiency is sufficiently large, i.e., $\varepsilon_t \geq \varepsilon_t^*$.

In Appendix B.2, we give necessary and sufficient conditions for the existence of bubbleless and bubbly steady states. For general distributions, there is no analytical solution for the steady states. We thus consider a special distribution, $\Phi(\varepsilon) = 1 - \varepsilon^{-\sigma}$ for $\varepsilon \geq 1$ and $\sigma > 1$. Suppose that $\xi > \delta(\sigma - 1)/(\pi\sigma)$. We find that both a bubbly equilibrium and a bubbleless equilibrium exist if $\delta > \sigma r$, and only a bubbleless equilibrium exists if $\delta < \sigma r$.

8 Stochastic Bubbles and Policy Implications

So far, we have focused on deterministic bubbles. Following Blanchard and Watson (1982) and Weil (1987), we now introduce stochastic bubbles to the baseline model in Section 2 with intratemporal loans. Suppose a bubble exists initially, i.e., $B_0 > 0$. In each time interval between t and $t + dt$, there is a probability θdt that the bubble will burst, i.e., $B_{t+dt} = 0$. Once it bursts, it will never be valued again so that $B_\tau = 0$ for all $\tau \geq t + dt$.[22] With the remaining probability $1 - \theta dt$, the bubble persists so that $B_{t+dt} > 0$. Later, we will take the continuous time limits as $dt \to 0$.

8.1 Equilibrium with Stochastic Bubbles

First, we consider the case in which the bubble has collapsed. This corresponds to the bubbleless equilibrium studied in Section 4. We use a variable with an asterisk (except for K_t) to denote its value in the bubbleless equilibrium. In particular, $V_t^*(K_t^j)$ denotes firm j's value function. In the continuous-time limit, (Q_t^*, K_t) satisfies the equilibrium system (20) and (21) with $B_t = 0$. We may express the solution for Q_t^* in a feedback form in that $Q_t^* = g(K_t)$ for some function g.

Next, we consider the case in which the bubble has not collapsed. We assume that the debt contract is not contingent on extraneous beliefs or sunspots described earlier. Firms borrow only when an investment opportunity arrives. The threat value to the lender or the stock market value of the collateralized assets is equal to the ex ante value before the realization of a sunspot, $e^{-rdt}V_{t+dt}(\xi K_t^j)(1 - \theta dt) + e^{-rdt}V_{t+dt}^*(\xi K_t^j)\theta dt$. Thus, the credit constraint is given

[22] If a bubble reemerged in the future, it would have value today by the no-arbitrage asset-pricing equation. To generate recurrent bubbles and crashes, Miao and Wang (2011b) introduce firm entry and exit in their model. See Martin and Ventura (2010a) and Wang and Wen (2011) for other approaches.

by

$$L_t^j \leq e^{-rdt} V_{t+dt}(\xi K_t^j)\,(1-\theta dt) + e^{-rdt} V_{t+dt}^*(\xi K_t^j)\theta dt. \tag{63}$$

We write firm j's dynamic programming problem as follows:

$$\begin{aligned} V_t(K_t^j) \;\; = \;\; & \max_{I_t^j} \; R_t K_t^j dt - \pi I_t^j dt \\ & + e^{-rdt}\,(1-\theta dt)\, V_{t+dt}((1-\delta dt)K_t^j + I_t^j)\pi dt \\ & + e^{-rdt}\,(1-\theta dt)\, V_{t+dt}((1-\delta dt)K_t^j)\,(1-\pi dt) \\ & + e^{-rdt}\theta dt\;\, V_{t+dt}^*((1-\delta dt)K_t^j + I_t^j)\pi dt \\ & + e^{-rdt}\theta dt\;\, V_{t+dt}^*((1-\delta dt)K_t^j)\,(1-\pi dt)\,, \end{aligned} \tag{64}$$

subject to (5) and (63).

We conjecture that the value function takes the form, $V_t(K_t^j) = v_t K_t^j + b_t$, where v_t and b_t are to be determined and are independent of K_t^j. As we have shown in Section 4, when the bubble bursts, the value function is given by $V_t^*(K_t^j) = v_t^* K_t^j$. After substituting these two value functions into (64) and simplifying, the firm's dynamic programming problem becomes

$$v_t K_t^j + b_t = \max_{I_t^j} \; R_t K_t^j dt - \pi I_t^j dt + Q_t(1-\delta dt)K_t^j + Q_t \pi I_t^j dt + B_t, \tag{65}$$

subject to

$$0 \leq I_t^j \leq R_t K_t^j dt + Q_t \xi K_t^j + B_t, \tag{66}$$

where we define $Q_t^* = e^{-rdt} v_{t+dt}^*$,

$$Q_t = e^{-rdt}\left[(1-\theta dt)v_{t+dt} + \theta v_{t+dt}^* dt\right], \tag{67}$$

$$B_t = e^{-rdt}(1-\theta dt) b_{t+dt}. \tag{68}$$

Suppose $Q_t > 1$. Then the optimal investment level achieves the upper bound in (66). Substituting this investment level into equation (65) and matching coefficients on the two sides of this equation, we obtain

$$v_t = R_t dt + Q_t(1-\delta dt) + \pi(Q_t - 1)(R_t dt + Q_t \xi)dt, \tag{69}$$

$$b_t = B_t + \pi(Q_t - 1)B_t dt. \tag{70}$$

As in Section 3, we conduct aggregation to obtain the discrete-time equilibrium system. We then take the continuous-time limits as $dt \to 0$ to obtain the following:

Proposition 10 *Suppose $Q_t > 1$. Before the bubble bursts, the equilibrium with stochastic bubbles (B_t, Q_t, K_t) satisfies the following system of differential equations:*

$$\dot{B}_t = (r + \theta)B_t - \pi(Q_t - 1)B_t, \tag{71}$$

$$\dot{Q}_t = (r + \delta + \theta)Q_t - \theta Q_t^* - R_t - \pi(Q_t - 1)\xi Q_t, \tag{72}$$

and (21), where $R_t = \alpha K_t^{\alpha-1}$ and $Q_t^* = g(K_t)$ is the capital price after the bubble bursts.

Equation (71) reveals that the expected rate of return on bubbles is equal to the interest rate r. In general, it is difficult to characterize the equilibrium with stochastic bubbles. In order to transparently illustrate the adverse impact of bubble bursting on the economy, we shall consider a simple type of equilibrium. Following Weil (1987) and Kocherlakota (2009), we study a stationary equilibrium with stochastic bubbles that has the following properties: The capital stock is constant at the value K_s over time before the bubble collapses. It continuously moves to the bubbleless steady-state value K^* after the bubble collapses. The bubble is also constant at the value $B_s > 0$ before it collapses. It jumps to zero and then stays at this value after collapsing. The capital price is constant at the value Q_s before the bubble collapses. It jumps to the value $g(K)$ after the bubble collapses and then converges to the bubbleless steady-state value Q^* given in equation (27).

Our objective is to show the existence of (B_s, Q_s, K_s). By (71), we can show that

$$Q_s = \frac{r + \theta}{\pi} + 1. \tag{73}$$

Since $Q_s > 1$, we can apply Proposition 10 in some neighborhood of Q_s. Equation (72) implies that

$$0 = (r + \delta + \theta)Q_s - \theta g(K) - R - \pi(Q_s - 1)\xi Q_s, \tag{74}$$

where $R = \alpha K^{\alpha-1}$. The solution to this equation gives K_s. Once we have obtained K_s and Q_s, we use equation (31) to determine B_s.

The difficult part is to solve for K_s since $g(K)$ is not an explicit function. To show the existence of K_s, we define θ^* as

$$\frac{r + \theta^*}{\pi} + 1 = \frac{\delta}{\pi\xi} = Q^*.$$

That is, θ^* is the bursting probability such that the capital price in the stationary equilibrium with stochastic bubbles is the same as that in the bubbleless equilibrium.

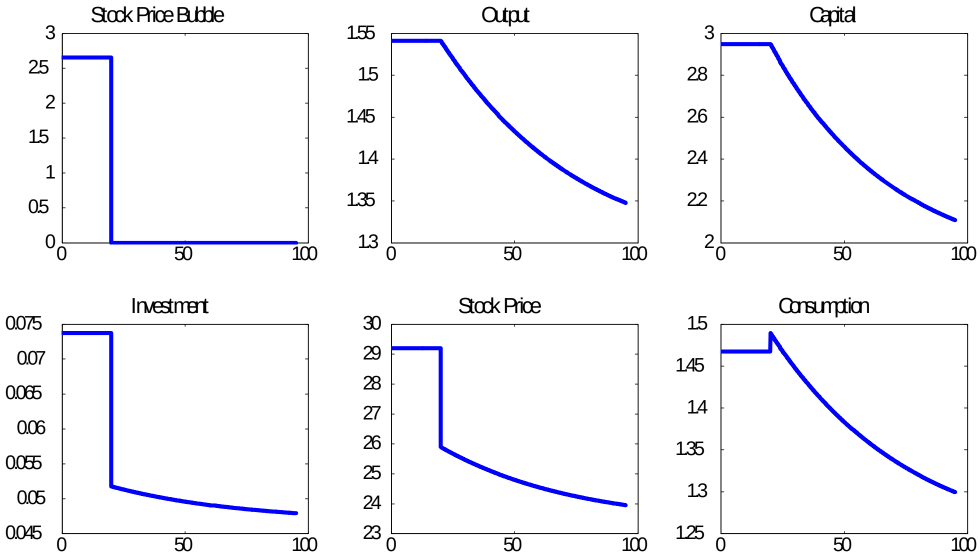

Figure 3: This figure plots the dynamics of the stationary equilibrium with stochastic bubbles. Assume that the bubble bursts at time $t = 20$. The parameter values are set as follows: $r = 0.02$, $\alpha = 0.4$, $\delta = 0.025$, $\theta = 0.05$, $\pi = 0.01$, and $\xi = 0.2$.

Proposition 11 *Let condition (35) hold. If $0 < \theta < \theta^*$, then there exists a stationary equilibrium (B_s, Q_s, K_s) with stochastic bubbles such that $K_s > K^*$. In addition, if θ is sufficiently small, then consumption falls eventually after the bubble bursts.*

As in Weil (1987), a stationary equilibrium with stochastic bubbles exists if the probability that the bubble will burst is sufficiently small. In Weil's (1987) overlapping generations model, the capital stock and output eventually rise after the bubble collapses. In contrast to his result, in our model the economy enters a recession after the bubble bursts in that consumption, capital and output all fall eventually. The intuition is that the collapse of the bubble tightens the collateral constraint and impairs investment efficiency.

Proposition 11 compares the economy before the bubble collapses with the economy after the bubble collapses only in the steady state. It would be interesting to see what happens along the transition path. Since analytical results are not available, we solve the transition path numerically and present the results in Figure 3.[23] In this numerical example, we assume that the bubble collapses at time $t = 20$. Immediately after the bubble collapses, investment falls discontinuously and then gradually decreases to its bubbleless steady-state level. But

[23] The parameter values for Figure 3 are not calibrated to match the data since the model is stylized. Miao and Wang (2011b) develop a quantitative DSGE model to study how asset bubbles can explain US business cycles.

output and capital decrease continuously to their bubbleless steady-state levels since capital is predetermined and labor is exogenous. Consumption rises initially because of the fall in investment. But it quickly falls and then decreases to its bubbleless steady-state level. Importantly, the stock market crashes immediately after the bubble collapses in that the stock price drops discontinuously. One way to generate the fall in consumption and output on impact is to introduce endogenous capacity utilization.[24] Following the collapse of bubbles, the capacity utilization rate falls because the price of installed capital rises. As a result, both output and consumption fall on impact.

8.2 Policy Implications

We have shown that the collapse of bubbles generates a recession. Is there a government policy that can restore economic efficiency? The inefficiency in our model comes from the credit constraints. Bubbles help relax these constraints, while the collapse of bubbles tightens them. Suppose that the government can supply liquidity to firms by issuing public bonds in the baseline model of Section 2. These bonds are backed by lump-sum taxes.[25] Firms can use public bonds as collateral to relax their credit constraints. They can also buy and sell these bonds to finance investment. Assume that firms and households face short-sales constraints (Kocherlakota (1992, 2009)). Without any trading frictions, bubbles are dominated by bonds which can also be used as collateral. As a result, a bubble cannot exist. Imposing market frictions, we shall show whether a government policy can eliminate a bubble and achieve efficiency.

Let the total issued quantity of government bonds be M_t and the bond price be P_t. We start with the discrete-time environment described in Section 2. The value of the government assets satisfies

$$M_t P_t = T_t dt + M_{t+dt} P_t,$$

where T_t denotes lump-sum taxes. Taking the continuous-time limit yields $\dot{M}_t P_t = -T_t$. Defining the government debt as $D_t = P_t M_t$, we can use the fact that $\dot{D}_t = \dot{P}_t M_t + \dot{M}_t P_t$ to derive

$$\dot{D}_t - \dot{P}_t M_t = \dot{D}_t - \frac{\dot{P}_t}{P_t} D_t = -T_t \text{ if } P_t > 0. \tag{75}$$

By a similar analysis to that in Section 6.2, we can derive that the price of the government

[24] Such an analysis is available upon request.

[25] As an idealized benchmark, we ignore the issues of moral hazard and distortional taxes.

bond satisfies the asset pricing equation:

$$rP_t = \dot{P}_t + \pi \left(Q_t - 1 \right) P_t.$$

If the government bond is not backed by taxes (i.e., $T_t = 0$), then it is a pure bubble and it can coexist with firm bubbles in equilibrium. As long as the government bond is backed by taxes (i.e., $T_t > 0$), then equation (75) implies that $\dot{D}_t = 0$ and $\dot{P}_t > 0$ in the steady state. Hence, it dominates firm bubbles since $\dot{B}_t = 0$ in the steady state. The government can then choose the amount of debt by adjusting the size of lump-sum taxes so that firms can overcome credit constraints.

Proposition 12 *Suppose assumption (35) holds. Let the government issue a constant value D of government debt given by*

$$D_t = D \equiv K_E \frac{\delta - \pi\xi}{\pi} > 0. \tag{76}$$

which is backed by lump-sum taxes $T_t = T \equiv rD$ for all t. Firms use government bonds as collateral. Then this credit policy will eliminate the bubble in firm value and enable the economy to achieve efficient allocation.

This proposition indicates that the government can design a policy that eliminates bubbles and achieves efficient allocation. The key intuition is that the government may provide sufficient liquidity to firms so that firms do not need to rely on bubbles to relax credit constraints. The government plays the role of financial intermediaries by transferring funds from households to firms directly so that firms can overcome credit constraints. The government bond is a store of value and can also generate dividends for firms. The collateral yield comes from the net benefit from new investment. Households prefer to sell bonds to firms as much as possible, but they face short-sales constraints. Under the government policy in the proposition, the government can make the growth rate of the bond price or the rate of capital gains exactly equal to the interest rate r. As a result, the dividend yield generated by the government bond is equal to zero, causing Tobin's marginal Q to equal 1.

To implement the above policy, the government constantly retires the public bonds at the interest rate in order to keep the total bond value constant.[26] To back the government bonds, the government levies constant lump-sum taxes equal to the interest payments of bonds.

[26] This policy is analogous to the Friedman rule in monetary economics.

9 Conclusion

In this paper, we provide an infinite-horizon model of a production economy with bubbles, in which firms meet stochastic investment opportunities and face endogenous credit constraints. Firms have limited commitment to repay debt. Credit constraints ensure that default never occurs in equilibrium. We show that bubbles can exist in firm value and help relax credit constraints and improve investment efficiency. This result holds for several types of endogenous credit constraints with limited commitment. We also introduce several types of assets and study the conditions under which firm bubbles can coexist with these types of assets. We show that firm bubbles and pure bubbles on intrinsically useless assets are perfect substitutes. The collapse of bubbles leads to a recession, even though there is no exogenous shock to the fundamentals of the economy. Immediately after the collapse, investment falls discontinuously and the stock market crashes in that the stock price falls discontinuously. In the long run, output, investment, consumption, and capital all fall to their bubbleless steady-state values. We show that there is a government policy that can eliminate the bubble in firm value and achieve efficient allocation.

We focus on firms' credit constraints and consider a deterministic economy in which firms are publicly traded in a stock market. Our analysis provides a theory of the creation and collapse of stock price bubbles driven by the credit market conditions. Our analysis differs from most studies in the literature that analyze bubbles on intrinsically useless assets or on assets with exogenously given rents or dividends in a pure exchange economy framework or an overlapping generations framework. Our model can incorporate this type of bubbles and thus provides a unified framework to study asset bubbles with firm heterogeneity and borrowing constraints. In future research, it would be interesting to consider households' endogenous borrowing constraints or incomplete markets economies and then study the role of bubbles in this kind of environment. It would also be interesting to study how bubbles contribute to business cycles in a quantitative dynamic stochastic general equilibrium model (see Miao, Wang and Xu (2012b)). Finally, there is no endogenous economic growth in the present paper. Miao and Wang (2011) extend the present paper to study endogenous growth.[27]

[27] See Grossman and Yanagawa (1993), Olivier (2000), Caballero, Farhi and Hammour (2006), Hirano and Yanagawa (2010) and Martin and Venture (2012) for models of asset bubbles and economic growth.

Appendices

A Proofs

Proof of Proposition 1: Substituting the conjecture (10) into (4) and (6) yields:

$$\begin{aligned} v_t K_t^j + b_t &= \max\ R_t K_t^j dt - \pi I_t^j dt + e^{-rdt} v_{t+dt} \left((1-\delta dt) K_t^j + I_t^j \right) \pi dt \\ &\quad + e^{-rdt} v_{t+dt} (1-\delta dt) K_t^j (1-\pi dt) + B_t, \end{aligned} \tag{A.1}$$

$$L_t^j \leq \xi e^{-rdt} v_{t+dt} K_t^j + B_t, \tag{A.2}$$

where B_t is defined in (12). We combine (5) and (A.2) to obtain

$$0 \leq I_t^j \leq R_t K_t^j dt + \xi e^{-rdt} v_{t+dt} K_t^j + B_t. \tag{A.3}$$

Let Q_t be the Lagrange multiplier associated with (3) for the case with the arrival of the investment opportunity. The first-order condition with respect to K_{t+dt}^j delivers equation (13). When $Q_t > 1$, we obtain the optimal investment rule in (11). Plugging (11) and (3) into the Bellman equation (A.1) and matching coefficients of K_t^j and the terms unrelated to K_t^j, we obtain (14) and (15). Q.E.D.

Proof of Proposition 2: Using the optimal investment rule in (11) and aggregating equation (3), we obtain the aggregate capital accumulation equation (18) and the aggregate investment equation (22) by a law of large numbers. Substituting (15) into (12) yields (16). Substituting (14) into (13) yields (17). The first-order condition for the static labor choice problem (1) gives $w_t = (1-\alpha)(K_t^j/N_t^j)^\alpha$. We then obtain (2) and $K_t^j = N_t^j \left(w_t/(1-\alpha)\right)^{1/\alpha}$. Thus, the capital-labor ratio is identical for each firm. Aggregating yields $K_t = N_t \left(w_t/(1-\alpha)\right)^{1/\alpha}$. Using this equation to substitute out w_t in (2) yields $R_t = \alpha K_t^{\alpha-1} N_t^{1-\alpha} = \alpha K_t^{\alpha-1}$ since $N_t = 1$ in equilibrium. Aggregate output satisfies

$$Y_t = \int (K_t^j)^\alpha (N_t^j)^{1-\alpha} dj = \int (K_t^j/N_t^j)^\alpha N_t^j dj = (K_t^j/N_t^j)^\alpha \int N_t^j dj = K_t^\alpha N_t^{1-\alpha}.$$

This completes the proof. Q.E.D.

Proof of Proposition 3: In the proof below, we drop all terms of order higher than dt. By equation (18),

$$\frac{K_{t+dt} - K_t}{dt} = -\delta K_t + [\xi Q_t K_t + B_t]\pi.$$

Taking the limit as $dt \to 0$ yields equation (21). Using the approximation $e^{rdt} = 1 + rdt$ in equation (16) yields:

$$B_t(1 + rdt) = B_{t+dt}\left[1 + \pi(Q_{t+dt} - 1)dt\right].$$

Simplifying yields:

$$\frac{B_t - B_{t+dt}}{dt} + rB_t = B_{t+dt}\pi(Q_{t+dt} - 1).$$

Taking the limits as $dt \to 0$ yields equation (19). Finally, we approximate equation (17) by

$$Q_t(1 + rdt) = R_{t+dt}dt + (1 - \delta dt)Q_{t+dt} + \xi Q_{t+dt}\left(Q_{t+dt} - 1\right)\pi dt.$$

Simplifying yields:

$$\frac{Q_t - Q_{t+dt}}{dt} + rQ_t = R_{t+dt} - \delta Q_{t+dt} + \xi Q_{t+dt}\left(Q_{t+dt} - 1\right)\pi.$$

Taking the limit as $dt \to 0$ yields equation (20).

We may start with a continuous-time formulation directly. Let $V\left(K_t^j, Q_t, B_t\right)$ denote the value function. The Bellman equation in continuous time is given by

$$\begin{aligned} rV\left(K_t^j, Q_t, B_t\right) &= \max_{I_t^j} R_t K_t^j + \pi\left[V\left(K_t^j + I_t^j, Q_t, B_t\right) - V\left(K_t^j, Q_t, B_t\right) - I_t^j\right] \\ &\quad -\delta K_t^j V_K\left(K_t^j, Q_t, B_t\right) + V_Q\left(K_t^j, Q_t, B_t\right)\dot{Q}_t + V_B\left(K_t^j, Q_t, B_t\right)\dot{B}_t, \end{aligned}$$

subject to $I_t^j \le V\left(\xi K_t^j, Q_t, B_t\right)$, where V_K, V_Q, and V_B represent partial derivatives. We may derive this Bellman equation by taking the limit in (4) as $dt \to 0$. Conjecture $V\left(K_t^j, Q_t, B_t\right) = Q_t K_t^j + B_t$. We can then solve the above Bellman equation. After aggregation, we can derive the system of differential equations in the proposition. Q.E.D.

Proof of Proposition 4: (i) The social planner solves the following problem:

$$\max_{I_t} \int_0^\infty e^{-rt}\left(K_t^\alpha - \pi I_t\right)dt,$$

subject to

$$\dot{K}_t = -\delta K_t + \pi I_t,\ K_0 \text{ given},$$

where K_t is the aggregate capital stock and I_t is the investment level for a firm with an investment opportunity. From this problem, we can derive the efficient capital stock K_E, which satisfies $\alpha\left(K_E\right)^{\alpha-1} = r + \delta$. The efficient output, investment and consumption levels are given by $Y_E = \left(K_E\right)^\alpha$, $I_E = \delta/\pi K_E$, and $C_E = \left(K_E\right)^\alpha - \delta K_E$, respectively.

Suppose assumption (25) holds. We conjecture that $Q^* = Q_t = 1$ in the steady state. In this case, firm value is given by $V\left(K_t^j\right) = K_t^j$. The optimal investment rule for each firm satisfies $R_t = r + \delta = \alpha K_t^{*\alpha-1}$. Thus, $K_t^* = K_E$. Given this constant capital stock for all firms, we must have $\delta K_t^* = \pi I_t^*$ for all t. Let each firm's optimal investment level satisfy $I_t^j = \delta K_t^j/\pi$. Then, when assumption (25) holds, the investment and credit constraints, $I_t^j = \delta K_t^j/\pi \leq \xi K_t^j = V\left(\xi K_t^j\right)$, are satisfied for all t . We conclude that, under assumption (25), the solutions $Q_t = 1$, $K_t^* = K_E$, and $I_t^*/K_t^* = \delta/\pi$ give the bubbleless equilibrium, which also delivers the efficient allocation.

(ii) Suppose (26) holds. Conjecture $Q_t > 1$ in some neighborhood of the bubbleless steady state. We can then apply Proposition 3 and derive the steady-state equations (23) and (24). From these equations, we obtain the steady-state solutions Q^* and K^* in (27) and (28), respectively. Assumption (26) implies that $Q^* > 1$. By continuity, $Q_t > 1$ in some neighborhood of (Q^*, K^*). This verifies our conjecture. Q.E.D.

Proof of Proposition 5: Solving equations (23), (30), and (31) yields equations (32)-(34). By (32), $B > 0$ if and only if (35) holds. From (27) and (33), we deduce that $Q_b < Q^*$. Using condition (35), it is straightforward to check that $K_{GR} > K_E > K_b > K^*$. From (32), it is also straightforward to verify that the bubble-asset ratio B/K_b decreases with ξ. Q.E.D.

Proof of Proposition 6: First, we consider the log-linearized system around the bubbly steady state (B, Q_b, K_b). We use $\hat{X}_t$ to denote the percentage deviation from the steady state value for any variable X_t, i.e., $\hat{X}_t = \ln X_t - \ln X$. We can show that the log-linearized system is given by

$$\begin{bmatrix} d\hat{B}_t/dt \\ d\hat{Q}_t/dt \\ d\hat{K}_t/dt \end{bmatrix} = A \begin{bmatrix} \hat{B}_t \\ \hat{Q}_t \\ \hat{K}_t \end{bmatrix},$$

where

$$A = \begin{bmatrix} 0 & -(r+\pi) & 0 \\ 0 & \delta + r - \xi(2r+\pi) & [(1-\xi)r+\delta](1-\alpha) \\ \pi B/K_b & \xi(r+\pi) & -\pi B/K_b \end{bmatrix}. \tag{A.4}$$

We denote this matrix by

$$A = \begin{bmatrix} 0 & a & 0 \\ 0 & b & c \\ d & e & f \end{bmatrix},$$

where we deduce from (A.4) that $a < 0$, $c > 0$, $d > 0$, $e > 0$, and $f < 0$. Since $\xi < \frac{\delta}{r+\pi}$, we have $b = (1-\xi)r + \delta - \xi(r+\pi) > 0$. The characteristic equation for the matrix A is

$$F(x) \equiv x^3 - (b+f)x^2 + (bf - ce)x - acd = 0. \tag{A.5}$$

We observe that $F(0) = -acd > 0$ and $F(-\infty) = -\infty$. Thus, there exists a negative root to the above equation, denoted by $\lambda_1 < 0$. Let the other two roots be λ_2 and λ_3. We rewrite $F(x)$ as

$$\begin{aligned} F(x) &= (x-\lambda_1)(x-\lambda_2)(x-\lambda_3) \\ &= x^3 - (\lambda_1+\lambda_2+\lambda_3)x^2 + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)x - \lambda_1\lambda_2\lambda_3. \end{aligned} \tag{A.6}$$

Matching terms in equations (A.5) and (A.6) yields $\lambda_1\lambda_2\lambda_3 = acd < 0$ and

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = bf - cd < 0. \tag{A.7}$$

We consider two cases. (i) If λ_2 and λ_3 are two real roots, then it follows from $\lambda_1 < 0$ that λ_2 and λ_3 must have the same sign. Suppose $\lambda_2 < 0$ and $\lambda_3 < 0$. We then have $\lambda_1\lambda_2 > 0$ and $\lambda_1\lambda_3 > 0$. This implies that $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 > 0$, which contradicts equation (A.7). Thus, we must have $\lambda_2 > 0$ and $\lambda_3 > 0$.

(ii) If either λ_2 or λ_3 is complex, then the other must also be complex. Let

$$\lambda_2 = g + hi \text{ and } \lambda_3 = g - hi,$$

where g and h are some real numbers. We can show that

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 2g\lambda_1 + g^2 + h^2.$$

Since $\lambda_1 < 0$, the above equation and equation (A.7) imply that $g > 0$.

From the above analysis, we conclude that the matrix A has one negative eigenvalue and the other two eigenvalues are either positive real numbers or complex numbers with a positive real part. As a result, the bubbly steady state is a local saddle point and the stable manifold is one dimensional.

Next, we consider the local dynamics around the bubbleless steady state $(0, Q^*, K^*)$. We linearize B_t around zero and log-linearize Q_t and K_t and obtain the following linearized system:

$$\begin{bmatrix} dB_t/dt \\ d\hat{Q}_t/dt \\ d\hat{K}_t/dt \end{bmatrix} = J \begin{bmatrix} B_t \\ \hat{Q}_t \\ \hat{K}_t \end{bmatrix},$$

where

$$J=\left[\begin{array}{ccc} r-\pi(Q^*-1) & 0 & 0 \\ 0 & a & b \\ \frac{\pi}{K^*} & c & d \end{array}\right],$$

where

$$\begin{aligned} a &= \frac{R^*}{Q^*}-\xi\pi Q^*, \\ b &= \frac{R^*}{Q^*}(1-\alpha)>0, \\ c &= \pi\xi Q^*>0, \\ d &= 0. \end{aligned}$$

Using a similar method for the bubbly steady state, we analyze the three eigenvalues of the matrix J. One eigenvalue, denoted by λ_1, is equal to $r-\pi(Q^*-1)<0$ and the other two, denoted by λ_2 and λ_3, satisfy

$$\lambda_2\lambda_3=ad-bc=0-bc<0. \tag{A.8}$$

It follows from (A.8) that λ_2 and λ_3 must be two real numbers with opposite signs. We conclude that the bubbleless steady state is a local saddle point and the stable manifold is two dimensional. Q.E.D.

Proof of Proposition 7: We work with the continuous-time limit. Let $V\left(K_t^j, L_t^j, B_t, Q_t\right)$ denote the value function. The continuous-time limit of the Bellman equation (36) is given by

$$\begin{aligned} & rV\left(K_t^j, L_t^j, B_t, Q_t\right) \\ = {} & \max_{D_t^j\geq 0}\left\{D_t^j+\dot{L}_t^j V_L\left(K_t^j, L_t^j, B_t, Q_t\right)\right\}-\delta K_t^j V_K\left(K_t^j, L_t^j, B_t, Q_t\right) \\ & +\pi\max_{L_1^j, I^j}\left\{V\left(K_t^j+I_t^j, L_{1t}^j, B_t, Q_t\right)+L_{1t}^j-I_t^j-L_t^j-V\left(K_t^j, L_t^j, B_t, Q_t\right)\right\} \\ & +\dot{B}_t V_B\left(K_t^j, L_t^j, B_t, Q_t\right)+\dot{Q}_t V_Q\left(K_t^j, L_t^j, B_t, Q_t\right), \end{aligned}$$

where the first max operator is subject to the constraint

$$\dot{L}_t^j=r_{ft}L_t^j+D_t^j-R_tK_t,$$

and the second max operator is subject to the constraints

$$I_t^j\leq L_{1t}^j-L_t^j, \tag{A.9}$$

$$V\left(K_t^j+I_t^j,L_{1t}^j,B_t,Q_t\right)\geq V\left(K_t^j+I_t^j,0,B_t,Q_t\right)-V\left(\xi\left(K_t^j+I_t^j\right),0,B_t,Q_t\right). \tag{A.10}$$

Note that when an investment opportunity arrives at a Poisson rate, the capital stock and firm debt jump.

Conjecture that

$$V\left(K_t^j,L_t^j,B_t,Q_t\right)=Q_tK_t^j-L_t^j+B_t.$$

Then, the credit constraint (A.10) becomes

$$Q_t\left(K_t^j+I_t^j\right)-L_{1t}^j+B_t\geq Q_t\left(K_t^j+I_t^j\right)+B_t-\xi Q_t\left(K_t^j+I_t^j\right)-B_t,$$

or

$$L_{1t}^j\leq \xi Q_t\left(K_t^j+I_t^j\right)+B_t. \tag{A.11}$$

Substituting the conjectured value function into the Bellman equation yields:

$$\begin{aligned} & rQ_tK_t^j-rL_t^j+rB_t \\ = & \max_{D_t^j\geq 0}\left\{D_t^j-\left(r_{ft}L_t^j+D_t^j-R_tK_t^j\right)\right\}-Q_t\delta K_t^j \\ & +\pi\max_{L_{1t}^j,I_t^j}\ \left(Q_t-1\right)I_t^j+\dot{B}_t+K_t^j\dot{Q}_t. \end{aligned}$$

It follows from (A.9) and (A.11) that

$$I_t^j\leq \xi Q_t\left(K_t^j+I_t^j\right)+B_t-L_t^j.$$

By assumption $Q_t<1/\xi$,

$$I_t^j\leq\frac{\xi Q_tK_t^j+B_t-L_t^j}{1-\xi Q_t}.$$

By assumption $Q_t>1$, the optimal investment level must reach the above upper bound

$$I_t^j=\frac{\xi Q_tK_t^j+B_t-L_t^j}{1-\xi Q_t}.$$

Substituting this solution into the Bellman equation and matching coefficients of K^j, L^j and other terms, we obtain equations (43), (44), and (45).

Since $r_{ft}<r$, households want to sell corporate bonds until their short-sales constraints bind, i.e., $L_t^h=0$. Thus, the market-clearing condition for the corporate bonds becomes $\int L_t^jdj=0$. By a law of large numbers, we can derive the law of motion for aggregate capital:

$$\begin{aligned} \dot{K}_t &= -\delta K_t+\pi\int I_t^jdj \\ &= -\delta K_t+\pi\frac{\xi Q_tK_t+B_t-\int L_t^jdj}{1-\xi Q_t}. \end{aligned}$$

Using the market clearing condition, $\int L_t^j dj = 0$, we obtain (42). In equilibrium, firms with investment opportunities choose to borrow and invest. For the bond market to clear, firms without investment opportunities must save and lend. Q.E.D.

Proof of Proposition 8: The proof is a straightforward application of Proposition 7. So we omit it here. Q.E.D.

Proof of Proposition 9: We work with the continuous-time limit. Let $V\left(K_t^j, M_t^j, B_t, Q_t, P_t\right)$ denote the value function. The continuous-time limit of the Bellman equation (36) is given by

$$
\begin{aligned}
& rV\left(K_t^j, M_t^j, B_t, Q_t, P_t\right) \\
= & \max_{D_t^j \geq 0} \left\{ D_t^j + \dot{M}_t^j V_M\left(K_t^j, M_t^j, B_t, Q_t, P_t\right)\right\} - \delta K_t^j V_K\left(K_t^j, M_t^j, B_t, Q_t, P_t\right) \\
& +\pi \max_{I_t^j, M_{1t}^j \geq 0} \left\{ V\left(K_t^j + I_t^j, M_{1t}^j, B_t, Q_t, P_t\right) + P_t M_t^j - I_t^j - P_t M_{1t}^j - V\left(K_t^j, M_t^j, B_t, Q_t, P_t\right)\right\} \\
& +\dot{B}_t V_B\left(K_t^j, M_t^j, B_t, Q_t, P_t\right) + \dot{Q}_t V_Q\left(K_t^j, M_t^j, B_t, Q_t, P_t\right) + \dot{P}_t V_P\left(K_t^j, M_t^j, B_t, Q_t, P_t\right),
\end{aligned}
$$

where the first max operator is subject to the constraint

$$D_t^j + P_t \dot{M}_t = R_t K_t + X M_t,$$

and the second max operator is subject to the constraints

$$I_t^j + P_t M_{1t}^j = L_t^j + P_t M_t^j, \tag{A.12}$$

$$L_t^j \leq V\left(\xi K_t^j, 0, B_t, Q_t\right). \tag{A.13}$$

Note that when an investment opportunity arrives at a Poisson rate, the capital stock and asset holdings jump.

Conjecture that

$$V\left(K_t^j, M_t^j, B_t, Q_t, P_t\right) = Q_t K_t^j + P_t M_t^j + B_t.$$

Then, the credit constraint (A.13) becomes

$$L_t^j \leq \xi Q_t K_t^j + B_t.$$

It follows from (A.12) that

$$I_t^j \leq \xi Q_t K_t^j + P_t M_t^j + B_t - P_t M_{1t}^j. \tag{A.14}$$

Notice that this constraint is the same as the continuous-time limit of (50) and (51).

Substituting the conjectured value function into the Bellman equation yields:

$$\begin{aligned} & r\left(Q_t K_t^j + P_t M_t^j + B_t\right) \\ = \; & R_t K_t + X M_t - \delta Q_t K_t + \pi \left(Q_t - 1\right) I_t^j + \dot{Q}_t K_t + \dot{B}_t + \dot{P}_t M_t^j. \end{aligned} \tag{A.15}$$

Under the assumption $Q_t > 1$, we must have $M_{1t}^j = 0$ and

$$I_t^j = \xi Q_t K_t^j + P_t M_t^j + B_t.$$

Substituting this solution into (A.15) and matching coefficients, we obtain equations (53), (54), and (55).

It follows from (55) that $rP_t > \dot{P}_t + X$. Thus, households will not hold the asset and their short-sales constraint binds, i.e., $M_t^h = 0$. This means that the market-clearing condition for the asset is given by $\int M_t^j dj = 1$.

By a law of large numbers, aggregate capital satisfies

$$\dot{K}_t = \delta K_t + \pi \left(\xi Q_t K_t + P_t \int M_t^j dj + B_t\right).$$

By the market-clearing condition, $\int M^j dj = 1$, we obtain (52). Q.E.D.

Proof of Proposition 10: As discussed in the main text, we may derive equations (69) and (70). Substituting equation (69) into (67) and using the definition $Q_t^* = e^{-rdt} v_{t+dt}^*$, we can derive that

$$\begin{aligned} Q_t \; = \; & \theta Q_t^* dt + e^{-rdt}(1 - \theta dt)[R_{t+dt} dt + Q_{t+dt}(1 - \delta dt) \\ & + \pi (Q_{t+dt} - 1)(R_{t+dt} dt + Q_{t+dt}\xi) dt]. \end{aligned} \tag{A.16}$$

Using the approximation $e^{-rdt} = 1 - rdt$ and removing all terms that have orders at least dt^2, we approximate the above equation by

$$\begin{aligned} Q_t - Q_{t+dt} \; = \; & \theta Q_t^* dt + R_{t+dt} dt - \delta Q_{t+dt} dt + \pi (Q_{t+dt} - 1)\xi Q_{t+dt} dt \\ & - (r + \theta)\, Q_{t+dt} dt. \end{aligned} \tag{A.17}$$

Dividing by dt on the two sides and taking the limit as $dt \to 0$, we obtain

$$-\dot{Q}_t = R_t - (r + \delta + \theta) Q_t + \theta Q_t^* + \pi (Q_t - 1)\xi Q_t, \tag{A.18}$$

which gives equation (72). Similarly, substituting equation (70) into (68) and taking the limit, we can derive equation (71).

We can also write down the continuous-time Bellman equation as follows:

$$\begin{aligned} rV\left(K_t^j, Q_t, B_t\right) &= R_t K_t^j + \pi \max_{I_t^j}\left[V\left(K_t^j + I_t^j, Q_t, B_t\right) - I_t^j - V\left(K_t^j, Q_t, B_t\right)\right] \\ &\quad +\theta\left[V\left(K_t^j, Q_t, B_t\right) - V^*\left(K_t^j, Q_t^*\right)\right] \\ &\quad -\delta K_t^j V\left(K_t^j, Q_t, B_t\right) + \dot{Q}_t V_Q\left(K_t^j, Q_t, B_t\right) + \dot{B}_t V_B\left(K_t^j, Q_t, B_t\right), \end{aligned}$$

which can be derived as the continuous-time limit of (64). Conjecture $V\left(K_t^j, Q_t, B_t\right) = Q_t K_t^j + B_t$ and $V^*\left(K_t^j, Q_t^*\right) = Q_t^* K_t^j$. We can derive Proposition 10. Q.E.D.

Proof of Proposition 11: Let $Q(\theta)$ be the expression on the right-hand side of equation (73). We then use this equation to rewrite equation (74) as

$$\alpha K^{\alpha-1} - (r+\delta+\theta)Q(\theta) + \theta g(K) + (r+\theta)\xi Q(\theta) = 0.$$

Define the function $F(K;\theta)$ as the expression on the left-hand side of the above equation. Notice $Q(\theta^*) = Q^* = g(K^*)$ by definition and $Q(0) = Q_b$ where Q_b is given in (33). The condition (35) ensures the existence of the bubbly steady-state value Q_b and the bubbleless steady-state values Q^* and K^*.

Define

$$K_{\max} = \max_{0\le\theta\le\theta^*}\left[\frac{(r+\delta+\theta-(r+\theta)\xi)Q(\theta) - \theta Q^*}{\alpha}\right]^{\frac{1}{\alpha-1}}.$$

By (34), we can show that

$$K_b = \left[\frac{(r+\delta-r\xi)Q(0)}{\alpha}\right]^{\frac{1}{\alpha-1}}.$$

Thus, we have $K_{\max} \ge K_b$ and hence $K_{\max} > K^*$. We want to prove that

$$F(K^*;\theta) > 0,\ F(K_{\max};\theta) < 0,$$

for $\theta \in (0,\theta^*)$. If this is true, then it follows from the intermediate value theorem that there exists a solution K_s to $F(K;\theta) = 0$ such that $K_s \in (K^*, K_{\max})$.

First, notice that

$$\begin{aligned} F(K^*, 0) &= \alpha K^{*\alpha-1} - r(1-\xi)Q_b - \delta Q_b \\ &> \alpha K_b^{\alpha-1} - r(1-\xi)Q_b - \delta Q_b = 0, \end{aligned}$$

and

$$F(K^*, \theta^*) = 0.$$

We can verify that $F(K;\theta)$ is concave in θ for any fixed K. Thus, for all $0 < \theta < \theta^*$,

$$\begin{aligned} F(K^*;\theta) &= F\left(K^*, (1-\frac{\theta}{\theta^*})0 + \frac{\theta}{\theta^*}\theta^*\right) \\ &> (1-\frac{\theta}{\theta^*})F(K^*,0) + \frac{\theta}{\theta^*}F(K^*,\theta^*) > 0. \end{aligned}$$

Next, for $K \in (K^*, K_{\max})$, we derive the following:

$$\begin{aligned} F(K_{\max};\theta) &= \alpha K_{\max}^{\alpha-1} - (r+\delta+\theta)Q(\theta) + \theta g(K_{\max}) + (r+\theta)\xi Q(\theta) \\ &< \alpha K_{\max}^{\alpha-1} - (r+\delta+\theta)Q(\theta) + \theta g(K^*) + (r+\theta)\xi Q(\theta) < 0, \end{aligned}$$

where the first inequality follows from the fact that the saddle path for the bubbleless equilibrium is downward sloping as illustrated in Figure 3 so that $g(K_{\max}) < g(K^*)$, and the second inequality follows from the definition of $K_{\max}$ and the fact that $g(K^*) = Q^*$.

Finally, note that $Q(\theta) < Q^*$ for $0 < \theta < \theta^*$. We use equation (31) and $K_s > K^*$ to deduce that

$$\frac{B_s}{K_s} = \frac{\delta}{\pi} - \xi Q(\theta) > \frac{\delta}{\pi} - \xi Q^* = 0.$$

This completes the proof of the existence of a stationary equilibrium with stochastic bubbles (B_s, Q_s, K_s).

When $\theta = 0$, the bubble never bursts and hence $K_s = K_b$. When θ is sufficiently small, K_s is close to K_b by continuity. Since K_b is smaller than the golden rule capital stock K_{GR}, $K_s < K_{GR}$ when θ is sufficiently small. Since $K^\alpha - \delta K$ is increasing for all $K < K_{GR}$, we deduce that $K_s^\alpha - \delta K_s > K^{*\alpha} - \delta K^*$. This implies that the consumption level before the bubble collapses is higher than the consumption level in the steady state after the bubble collapses. Q.E.D.

Proof of Proposition 12: The equilibrium with government debt can be characterized similarly to that in Proposition 9. In particular, (K_t, Q_t, B_t, P_t) satisfies

$$\dot{K}_t = -\delta K_t + \pi(Q_t \xi K_t + P_t M_t + B_t), \tag{A.19}$$

$$\dot{Q}_t = (r+\delta)Q_t - R_t - \pi(Q_t - 1)Q_t\xi, \tag{A.20}$$

$$\dot{B}_t = rB_t - \pi(Q_t - 1)B_t, \tag{A.21}$$

$$\dot{P}_t = rP_t - \pi(Q_t - 1)P_t, \tag{A.22}$$

and the usual transversality condition. The difference from Proposition 9 is that (i) the total supply of the government bond is M_t instead of 1, and (ii) $X = 0$.

Under the policy in the proposition, equation (75) implies that $\dot{P}_t = rP_t$. It follows from (A.22) that $Q_t = 1$. Substituting it into equation (A.20) reveals that $R_t = r + \delta$. This equation gives the efficient capital stock K_E for all time t after the collapse of the bubble. Let $K_t = K_E$ and $P_t M_t = D$ in (A.19), where D is given by (76). We can show that $B_t = 0$ for all t. Q.E.D.

B Alternative Credit Constraints

In this appendix, we study equilibrium in continuous time with two alternative credit constraints introduced in Section 7.

B. 1 Self-Enforcing Constraints

We study the continuous time case. First, consider the problem after default. It is a standard control problem and one can easily verify that the autarky value is given by $V_t^a\left(K_t^j\right) = Q_t^a K_t^j$, where Q_t^a satisfies the differential equation,

$$rQ_t^a = -\delta Q_t^a + R_t + \dot{Q}_t^a. \tag{B.1}$$

There is no bubble in the autarky value after a default by the transversality condition.

Next, consider the problem before default. Let $V\left(K_t^j, Q_t, B_t\right)$ denote the value function. Then it satisfies the continuous-time Bellman equation:

$$\begin{aligned} rV\left(K_t^j, Q_t, B_t\right) &= \max_{I^j} \ R_t K_t^j - \pi I_t^j + \pi\left[V\left(K_t^j + I_t^j, Q_t, B_t\right) - V\left(K_t^j, Q_t, B_t\right)\right] \\ &\quad -\delta K_t^j V_K\left(K_t^j, Q_t, B_t\right) + V_Q\left(K_t^j, Q_t, B_t\right)\dot{Q}_t + V_B\left(K_t^j, Q_t, B_t\right)\dot{B}_t, \end{aligned}$$

subject to the continuous-time limits of (5) and (57),

$$L_t^j \le V\left(K_t^j + I_t^j, Q_t, B_t\right) - V_t^a\left(K_t^j + I_t^j\right).$$

Conjecture that

$$V\left(K_t^j, Q_t, B_t\right) = Q_t K_t^j + B_t.$$

Substituting this conjecture and $V_t^a\left(K_t^j + I_t^j\right) = Q_t^a\left(K_t^j + I_t^j\right)$ into the above control problem, we obtain

$$r\left(Q_t K_t^j + B_t\right) = \max_{I_t^j} \ R_t K_t^j + \pi\left(Q_t - 1\right) I_t^j - \delta K_t^j Q_t + K_t^j \dot{Q}_t + \dot{B}_t,$$

subject to

$$0 \leq I_t^j \leq L_t^j, \tag{B.2}$$

$$L_t^j \leq (Q_t - Q_t^a)\left(K_t^j + I_t^j\right) + B_t. \tag{B.3}$$

If $Q_t > 1$ and $Q_t - Q_t^a < 1$, then the credit constraint binds and the optimal investment level is given by

$$I_t^j = \frac{(Q_t - Q_t^a)K_t^j + B_t}{1 - (Q_t - Q_t^a)}. \tag{B.4}$$

Substituting this equation back into the Bellman equation and matching coefficients, we can derive

$$rB_t = \dot{B}_t + \frac{\pi (Q_t - 1) B_t}{1 - (Q_t - Q_t^a)},$$

$$rQ_t = R_t + \pi (Q_t - 1) \frac{(Q_t - Q_t^a)}{1 - (Q_t - Q_t^a)} - \delta Q_t + \dot{Q}_t. \tag{B.5}$$

Comparing equations (B.1) and (B.5), we can see that if Q_t^a is a solution to (B.1), then $Q_t = Q_t^a$ is also a solution to (B.5). This solution makes economic sense since the marginal value of capital should not change immediately after default by no arbitrage. Setting $Q_t = Q_t^a$, the above two equations become

$$rB_t = \dot{B}_t + \pi (Q_t - 1) B_t,$$

$$rQ_t = R_t - \delta Q_t + \dot{Q}_t.$$

The credit constraint becomes $L_t^j = B_t$. Using (3) and (B.4) and setting $Q_t = Q_t^a$, we obtain the law of motion for aggregate capital:

$$\dot{K}_t = -\delta K_t + \pi B_t, \; K_0 \text{ given.}$$

The above three differential equations are identical to (19), (20), and (21) for $\xi = 0$. Thus, the analysis in Sections 4 and 5 for $\xi = 0$ applies here. Both bubbleless and bubbly equilibria exist and each type is unique. Q.E.D.

B. 2 Firm Value as Collateral

As in the proof of Proposition 3, we write the continuous-time Bellman equation as

$$\begin{aligned} rV\left(K_t^j, Q_t, B_t\right) = & \max_{I_t^j} \; R_t K_t^j + \pi \left\{ \int \left[V\left(K_t^j + I_t^j \varepsilon, Q_t, B_t\right) - I_t^j \right] d\Phi(\varepsilon) - V\left(K_t^j, Q_t, B_t\right) \right\} \\ & - \delta K_t^j V_K\left(K_t^j, Q_t, B_t\right) + V_Q\left(K_t^j, Q_t, B_t\right) \dot{Q}_t + V_B\left(K_t^j, Q_t, B_t\right) \dot{B}_t, \end{aligned}$$

subject to (5), (58), and (59). Conjecture that

$$V\left(K_t^j, Q_t, B_t\right) = Q_t K_t^j + B_t.$$

Substituting this conjecture into the Bellman equation yields:

$$\begin{aligned} r\left(Q_t K_t^j + B_t\right) &= \max_{I_t^j(\varepsilon)} R_t K_t^j + \pi\left[Q_t K_t^j + B_t + \int \left(Q_t\varepsilon - 1\right) I_t^j\left(\varepsilon\right) d\Phi\left(\varepsilon\right) - \left(Q_t K_t^j + B_t\right)\right] \\ &\quad -\delta K_t^j Q_t + \dot{Q}_t K_t^j + \dot{B}_t, \end{aligned}$$

where we make it explicit that investment I_t^j is a function of ε. The credit constraint (58) becomes

$$L_t^j \leq \xi\left(Q_t K_t^j + B_t\right).$$

Clearly, when $\varepsilon Q_t - 1 > 0$, firm j chooses to invest at the the maximal level such that the credit constraint binds. Thus, the optimal investment rule is given by

$$I_t^j(\varepsilon) = \left\{ \begin{array}{cc} \xi Q_t K_t^j + \xi B_t & \text{if } \varepsilon \geq \varepsilon^* \\ 0 & \text{otherwise} \end{array} \right\}, \tag{B.6}$$

where $\varepsilon^* = 1/Q_t$. Substituting this investment rule back into the Bellman equation and matching coefficients, we obtain equations (60) and (61). Using equation (59) and the above investment rule, we can derive the law of motion for aggregate capital stock in equation (62).

We first consider the bubbleless equilibrium in which $B_t = 0$. The bubbleless steady state (Q, K) is characterized by the following two equations:

$$rQ = -\delta Q + R + \pi\xi Q \int_{\frac{1}{Q}}^{\varepsilon_{\max}} (Q\varepsilon - 1) d\Phi(\varepsilon), \tag{B.7}$$

$$0 = -\delta K + \pi\xi Q K \int_{\frac{1}{Q}}^{\varepsilon_{\max}} \varepsilon d\Phi(\varepsilon). \tag{B.8}$$

Simplifying yields one equation for one unknown $\varepsilon^* = 1/Q$:

$$\xi \int_{\varepsilon^*}^{\varepsilon_{\max}} \varepsilon\phi(\varepsilon) d\varepsilon = \frac{\delta}{\pi}\varepsilon^*. \tag{B.9}$$

One can check that the expression on the left-hand side of the above equation is a decreasing function of ε^* and the expression on the right-hand side is an increasing function of ε^*. Thus, given $0 < \pi \leq 1$ and the assumption that

$$\xi \int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \varepsilon\phi(\varepsilon) d\varepsilon > \frac{\delta}{\pi}\varepsilon_{\min}, \tag{B.10}$$

there exists a unique interior solution $\varepsilon_f^* \in (\varepsilon_{\min}, \varepsilon_{\max})$ to equation (B.9) by the intermediate value theorem. The above condition is also necessary.

Now let us turn to the bubbly equilibrium in which $B_t > 0$ for all t. The bubbly steady state is characterized by three equations for three unknowns (B, Q, K) :

$$\pi\xi \int_{\frac{1}{Q}} (Q\varepsilon - 1) d\Phi(\varepsilon) = r, \tag{B.11}$$

$$(r + \delta)Q = R + \pi\xi Q \int_{\frac{1}{Q}} (Q\varepsilon - 1) d\Phi(\varepsilon), \tag{B.12}$$

$$\delta K = \pi(\xi QK + \xi B) \int_{\frac{1}{Q}} \varepsilon d\Phi(\varepsilon), \tag{B.13}$$

where $R = \alpha K^{\alpha-1}$.

We claim that there exists a bubbly steady state if and only if

$$\pi\xi \int_{\varepsilon_f^*}^{\varepsilon_{\max}} (\frac{\varepsilon}{\varepsilon_f^*} - 1) d\Phi(\varepsilon) > r. \tag{B.14}$$

To prove this claim, we define a function

$$G(x) = \pi\xi \int_{x}^{\varepsilon_{\max}} (\frac{\varepsilon}{x} - 1) d\Phi(\varepsilon).$$

It can be easily verified that G is a decreasing function of x. Since $G\left(\varepsilon_f^*\right) > r$ by assumption (B.14) and $G(\varepsilon_{\max}) = 0 < r$, there exists a unique solution $\varepsilon_b^* \in \left(\varepsilon_f^*, \varepsilon_{\max}\right)$ to the equation $G(\varepsilon_b^*) = r$ by the intermediate value theorem. Let $Q = 1/\varepsilon_b^*$. This means that condition (B.14) is a sufficient condition for the existence of Q in equation (B.11).

Once the bubbly steady state Q is determined, we turn to B/K. Using (B.12) and (B.13) and $Q = 1/\varepsilon_b^*$, we can solve for B/K :

$$\begin{aligned} \frac{B}{K} &= \frac{\delta}{\pi\xi} \frac{1}{\int_{\varepsilon_b^*} \varepsilon d\Phi(\varepsilon)} - \frac{1}{\varepsilon_b^*} \\ &= \frac{\delta\varepsilon_b^* - \pi\xi \int_{\varepsilon_b^*} \varepsilon d\Phi(\varepsilon)}{\pi\xi\varepsilon_b^* \int_{\varepsilon_b^*} \varepsilon d\Phi(\varepsilon)}. \end{aligned} \tag{B.15}$$

We need $B/K > 0$, or equivalently,

$$H(\varepsilon_b^*) \equiv \delta\varepsilon_b^* - \pi\xi \int_{\varepsilon_b^*} \varepsilon d\Phi(\varepsilon).$$

We can check that H is an increasing function of ε_b^*. In addition, $H\left(\varepsilon_f^*\right) = 0$ by equation (B.9). Since $\varepsilon_b^* > \varepsilon_f^*$, we deduce $H(\varepsilon_b^*) > 0$.

To prove the necessity of condition (B.14) for the existence of a bubbly steady state, we suppose that a bubbly steady state $\varepsilon_b^* = 1/Q$ satisfying (B.11) exists. Then $G(\varepsilon_b^*) = r$. Since $B/K > 0$, we have $H(\varepsilon_b^*) > 0$. But $H\left(\varepsilon_f^*\right) = 0$ by (B.9) and H is an increasing function. We deduce that $\varepsilon_b^* > \varepsilon_f^*$. Since G is a decreasing function, we conclude that $G\left(\varepsilon_f^*\right) > r$, which is condition (B.14).

Given the distribution $\Phi(\varepsilon) = 1 - \varepsilon^{-\sigma}$ for $\varepsilon \geq 1$ and $\sigma > 1$, we can easily verify that conditions (B.10) and (B.14) are equivalent to

$$\frac{\sigma}{\sigma - 1} > \frac{\delta}{\pi}\frac{1}{\xi}, \tag{B.16}$$

$$\sigma < \frac{\delta}{r}. \tag{B.17}$$

We can compute that the bubble is given by

$$B = \left(\frac{\delta}{\alpha}\right)^{\frac{1}{\alpha-1}} [r(\sigma-1)]^{\frac{1}{\sigma(\alpha-1)} - \frac{\sigma-1}{\sigma}} (\pi\xi)^{\frac{\alpha}{\sigma(1-\alpha)}} (\delta - r\sigma) > 0.$$

Q.E.D.

References

Albuquerque, Rui and Hugo A. Hopenhayn, 2004, Optimal Lending Contracts and Firm Dynamics, *Review of Economic Studies* 71, 285-315.

Alvarez, Fernando and Urban J. Jermann, 2000, Efficiency, Equilibrium, and Asset Pricing with Risk of Default, *Econometrica* 68, 775-798.

Benhabib, Jess and Farmer, Roger, 1999. Indeterminacy and Sunspots in Macroeconomics, in: Taylor, J., Woodford, M. (Eds.), *Handbook of Macroeconomics*, Vol. 1A, North-Holland, Amsterdam, 387–448.

Bernanke, Ben, and Mark Gertler, 1989, Agency Costs, Net Worth, and Business Fluctuations, *American Economic Review* 79(1), 14-31.

Bernanke, Ben, Mark Gertler and Simon Gilchrist, 1999, Financial Accelerator in a Quantitative Business Cycle Framework, In: J. Taylor and M. Woodford (eds.), *Handbook of Macroeconomics*, Elsevier Science, Amsterdam.

Blanchard, Olivier, and Mark Watson, 1982, Bubbles, Rational Expectations and Financial Markets, Harvard Institute of Economic Research Working Paper No. 945.

Brunnermeier, Markus, 2009, Bubbles, entry in *The New Palgrave Dictionary of Economics*, edited by Steven Durlauf and Lawrence Blume, 2nd edition.

Bulow, Jeremy and Kenneth Rogoff, 1989, Sovereign Debt: Is to Forgive to Forget?, *American Economic Review* 79, 43-50.

Burnside, Craig, Martin Eichenbaum and Sergio Rebelo, 2011, Understanding Booms and Busts in Housing Prices, working paper, Northwestern University.

Caballero, Ricardo, Emmanuel Farhi and Mohamad Hammour, 2006, Speculative Growth: Hints from the US Economy, *American Economic Review* 96, 1159-1192

Caballero, Ricardo J. and Arvind Krishnamurthy, 2006, Bubbles and Capital Flow Volatility: Causes and Risk Management, *Journal of Monetary Economics* 53, 35-53.

Carlstrom, Charles T. and Timothy S. Fuerst, 1997, Agency Costs, Net Worth, and Business Fluctuations: A Computable General Equilibrium Analysis, *American Economic Review* 87, 893-910.

Chaney, Thomas, David Sraer, and David Thesmar, 2009, The Collateral Channel: How Real Estate Shocks affect Corporate Investment, working paper, Princeton University.

Collyns, Charles and Abdelhak Senhadji, 2002, Lending Booms, Real Estate Bubbles, and The Asian Crisis, IMF Working Paper No. 02/20.

Diamond, Peter, 1965, National Debt in a Neoclassical Growth Model, *American Economic Review* 55, 1126-1150.

Farhi, Emmanuel and Jean Tirole, 2012, Bubbly Liquidity, *Review of Economic Studies* 79, 678-706.

Farmer, Roger E. A., 1999, *The Macroeconomics of Self-Fulfilling Prophecies*, 2nd ed., MIT Press, Cambridge.

Gan, Jie, 2007, Collateral, Debt Capacity, and Corporate Investment: Evidence from a Natural Experiment, *Journal of Financial Economics* 85, 709-734.

Goyal, Vidhan K. and Takeshi Yamada, 2004, Asset Price Shocks, Financial Constraints, and Investment: Evidence from Japan, *Journal of Business* 77, 175-200.

Grossman, Gene and Noriyuki Yanagawa, 1993, Asset Bubbles and Endogenous Growth, *Journal of Monetary Economics* 31, 3-19.

Gurkaynak, Refet S., 2008, Econometric Test of Asset Price Bubbles: Taking Stock, *Journal of Economic Surveys* 22, 166-186.

Hayashi, Fumio, 1982, Tobin's Marginal q and Average q: A Neoclassical Interpretation, *Econometrica* 50, 213-224.

Hellwig, Christian, and Guido Lorenzoni, 2009, Bubbles and Self-Enforcing Debt, *Econometrica* 77, 1137-1164.

Hirano, Tomohiro and Noriyuki Yanagawa, 2010, Asset Bubbles, Endogenous Growth and Financial Frictions, working paper, University of Tokyo.

Jermann, Urban, and Vincenzo Quadrini, 2012, Macroeconomic Effects of Financial Shocks, *American Economic Review* 102, 238-271.

Jovanovic, Boyan, 2013, Bubbles in Prices in Exhaustible Resources, *International Economic Review* 54, 1-34.

Kehoe, Timothy J. and David K. Levine, 1993, Debt-Constrained Asset Markets, *Review of Economic Studies* 60, 865-888.

Kiyotaki, Nobuhiro, and John Moore, 1997, Credit Cycles, *Journal of Political Economy* 105, 211-248.

Kiyotaki, Nobuhiro, and John Moore, 2005, Liquidity and Asset Prices, *International Economic Review* 46, 317-349.

Kiyotaki, Nobuhiro, and John Moore, 2008, Liquidity, Business Cycles and Monetary Policy, working paper, Princeton University.

Kocherlakota, Narayana, 1992, Bubbles and Constraints on Debt Accumulation, *Journal of Economic Theory* 57, 245-256.

Kocherlakota, Narayana, 2008, Injecting Rational Bubbles, *Journal of Economic Theory* 142, 218-232.

Kocherlakota, Narayana, 2009, Bursting Bubbles: Consequences and Cures, working paper, University of Minnesota.

Martin, Alberto and Jaume Ventura, 2011, Theoretical Notes on Bubbles and the Current Crisis, IMF Economic Review 59, 6-40.

Martin, Alberto and Jaume Ventura, 2012, Economic Growth with Bubbles, *American Economic Review* 102, 3033-3058.

Miao, Jianjun and Pengfei Wang, 2010, Credit Risk and Business Cycles, working paper, Boston University.

Miao, Jianjun and Pengfei Wang, 2011, Sectoral Bubbles and Endogenous Growth, working paper, Boston University.

Miao, Jianjun and Pengfei Wang, 2012a, Bubbles and Total Factor Productivity, *American Economic Review Papers and Proceedings* 102, 82-87.

Miao, Jianjun, Pengfei Wang, and Zhiwei Xu, 2012b, A Bayesian DSGE Model of Stock Market Bubbles and Business Cycles, working paper, Boston University.

Olivier, Jacques, 2000, Growth-Enhancing Bubbles, *International Economic Review* 41, 133-151.

Santos, Manuel S. and Michael Woodford, 1997, Rational Asset Pricing Bubbles, *Econometrica* 65, 19-58.

Scheinkman, Jose A. and Laurence Weiss, 1986, Borrowing Constraints and Aggregate Economic Activity, *Econometrica* 54, 23-46.

Scheinkman, Jose, A. and Wei Xiong, 2003, Overconfidence and Speculative Bubbles, *Journal of Political Economy* 111, 1183-1219.

Shiller, Robert, J., 2005, Irrational Exuberance, 2nd Edition, Princeton University Press.

Tirole, Jean, 1982, On the Possibility of Speculation under Rational Expectations, *Econometrica* 50, 1163-1182.

Tirole, Jean, 1985, Asset Bubbles and Overlapping Generations, *Econometrica* 53, 1499-1528.

Tirole, Jean, 2006, *The Theory of Corporate Finance*, Princeton University Press, Princeton and Oxford.

Tobin, James, 1969, A General Equilibrium Approach to Monetary Policy, *Journal of Money, Credit, and Banking* 1, 15-29.

Wang, Pengfei and Yi Wen, 2012, Speculative Bubbles and Financial Crisis, *American Economic Journal: Macroeconomics* 4, 184-221.

Wei, Xiong and Jialin Yu, 2011, The Chinese Warrants Bubble, *American Economic Review* 101, 2723-2753.

Weil, Philippe, 1987, Confidence and the Real Value of Money in an Overlapping Generations Economy, *Quarterly Journal of Economics* 102, 1-22.