Sequential Innovation and Patent Policy

Job Market Paper*

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Abstract

I use a model of sequential innovation with creative destruction to study how patent policy shapes R&D investment dynamics. I find that investments are non-stationary and increase as the patent's expiration date approaches. R&D investments are driven by the incremental rent that firms obtain from innovating. For entrants, this rent is the value of a new patent. For incumbents, the rent is the value of a new patent minus a loss of profits from replacing the currently active patent. The main finding is that, in a sequential world, patent policy not only affects the value of a new patent, but it also affects the replacement value of a currently active patent. As a consequence, an increase in patent length decreases the incumbent's investment rate and may decrease the economy's innovation rate. Incorporating this new observation, I study optimal patent policy and how it depends on different market characteristics, such as the market's natural innovation rate and the market's matureness.

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1 Introduction

Inventions and the development of new knowledge are at the heart of the progress of modern economies. Much of the microeconomic dynamics within markets are generated by temporary competitive advantages created by the introduction of new products or the adoption of new production processes. Patents are the main tool that governments use to promote innovation. By granting the right of exclusion, patents help firms to obtain enough profits to incentivize the pursuit of new discoveries. However, the incentives provided by a patent system are not yet fully understood. The goal of this paper is to answer the following questions on the effects of patent policy in a context of *sequential* innovation (i.e. where innovations build upon each other): 1) what are the *dynamic* incentives induced by patent policy on firms' R&D decisions? 2) Given these incentives, what are the key determinants of an optimal patent policy and how does patent policy relate to potentially observable market characteristics?

The classic work on patents (e.g. Nordhaus, 1969; Loury, 1979; and Lee and Wilde, 1980) treats each innovation as an isolated phenomenon, ignoring how the prospect of future breakthroughs affect the value of an innovation and, ultimately, the incentives to innovate at any given point in time. This literature sustains the idea that longer and more enforceable (stronger) patents increase the returns of a succeeding innovator. Consequently, the design of an optimal patent system consists only in assessing the benefit of higher innovation rates, induced by stronger patents, and the monopoly cost associated with them. As we shall see, in a sequential world this logic is not complete. Even though patents are necessary to induce R&D, patent systems that are "too strong" induce a relatively low innovation arrival rate in the economy, implying that the welfare losses of a strong patent policy lie beyond the loss in consumer surplus due to monopoly power.

In this paper, I provide a tractable continuous-time model of R&D investments in an economy where innovation is sequential and has the Schumpeterian property of creative destruction. Since patent protection is finite, the value of having a patent decreases as the patent expiration date approaches. This induces R&D incentives –and consequently, investment efforts– to be non-stationary. Also, it makes patent policy play an important role in the *timing* of R&D investments. At any point in time, the prize obtained from an innovation corresponds to the *incremental rent* derived from a new patent. For an entrant, this is simply the value of a new patent. Instead, for the incumbent, the incremental rent corresponds to the *difference* between the value of a new patent and the current value of his active patent. The key insight gained from studying the problem in a sequential context is that patent policy not only affects the value of a new innovation, it also affects the incumbent's valuation for his currently active patent.

The difference in valuation for a new innovation between incumbents and entrants leads them to react asymmetrically to changes in patent policy. For instance, an increase in patent length raises the value of a new patent, inducing entrants to increase their investment effort. In contrast, an increase in patent length leads incumbents not only to value a new patent more but also to increase their valuation for their currently active patent. I show that the increase in valuation of the current patent is larger than the increase in valuation for a new patent. This effect reduces the prize that an incumbent obtains from innovating, reducing his incremental rent and, ultimately, decreasing the incumbent's investment rate. As a consequence, when patent protection lasts for "too long" the decrease in the incumbent's investment rate overcomes the increase in the entrants investments, implying that the economy's innovation rate may decrease with an increase in patent protection.

After understanding the firms' investment dynamics, I study the welfare implications of a patent system. To do so, I introduce a representative consumer to the framework and interpret the sequential innovation model as a quality ladder growth model. Under this set up, I study how optimal patent policy varies with market characteristics, such as the market's natural innovation rate and the magnitude of the quality improvement. I also study the commitment incentives that a social planner faces. In particular, I compare the system that the planer would implement to incentivize the generation of quality improvements on an existing market, to the system that would be implemented in the context where the market has to be created.

This paper represents a contribution to the innovation literature in three respects. First, I provide a clean theoretical model that sheds light on how the dynamic incentives for innovation evolve through the patent life. To my knowledge, this is the first paper that, in a context of sequential innovation, i) studies the incentives induced by *finite* patent lengths, ii) looks at optimal patent policy, and iii) performs comparative statics. Secondly, my model rationalizes some empirical findings, such as incumbents: invest less than entrants (Czarnitzki and Kraft 2004, Acs and Audretsch 1988), patent at lower rates (Bound *et al.*, 1984), and, further, invest at a slower pace (Igami, 2011). Moreover, it is consistent with the empirical evidence provided by Sakakibara and Branstetter (2001) that stronger patent policy does not necessarily lead to more R&D (see Cohen 2010 for exhaustive survey in the empirical R&D literature). Lastly, my findings bring new insights and provide deeper understanding of patent systems –the most common tool used by governments to promote innovation.

The paper is organized as follows: Section 2 presents a simple model of innovation which is the basis of the subsequent analysis. In Section 3, I prove the equilibrium's existence and find its analytical solution. There, I show that the incumbent's R&D investments increase as the patent expiration date approaches. This increase in investments is driven by the fact that the commercialization of a new invention cannibalizes profits from the incumbent's active patent (this is also known as Arrow's replacement effect). As time goes by, the remaining value of the incumbent's active patent decreases to zero, increasing the marginal benefit of a new innovation, pushing investment up (Proposition 1).

In Section 4, as a comparative static analysis, I explore investment dynamics. I show that under longer patent protection the value of a new patent increases (Proposition 2). Despite the increase in the value of a new patent, firms react differently depending on whether they are the incumbent or an entrant (Proposition 5). Entrants' investments increase, as their incentives are aligned with the value of a new patent. In contrast, incumbents decrease their effort when they are offered longer protection. This decrease is caused by the fact that in a sequential context patent policy not only affects the value of a new innovation, it also affects the value of the active patent held by the incumbent. An increase in patent length increases the value of the active patent more than the increase in the value of a new patent, decreasing the incumbent's prize from innovating. The intuition for this result can be explained by the observation that the *effective* patent length generally differs from the *statutory* length. As a result, the incumbent is more likely to take advantage of the patent extension the closer he is to the patent expiration date. Consequently, the increase in the value of an active patent is greater the closer the patent is to its expiration date.

The reduction in the incumbent's effort rate causes the economy's innovation rate react non-monotonically to changes in patent length (Result 9). While with very short patents R&D is barely performed, patents that are "too long" not only reduce the incumbent's innovation rate, they may also decrease the economy's innovation rate (as demonstrated in Figure 4a, the relation is inverted-U shaped.)

I also study the effects of patent breadth in the firms' investment dynamics. I model patent breadth as the combination of two variables: the probability that an entrant infringes the currently active patent, and the magnitude of the compulsory license fees that he has to pay in such case. I assume that innovators have perfect protection from imitation and that all innovations are patentable. New innovations, however, may infringe the current leading technology with some positive probability. In the case that an infringement occurs, the infringing firm pays the infringed incumbent a compulsory license fee in order to undertake the incumbent's position. In general, I find that greater patent breadth increases the value of a new innovation and encourages incumbents to invest more (Result 6.2.) Also, I find that greater breadth deters entrants from innovating (Result 6.1.). The net effect on the total investments depends on the market's natural innovation rate. In particular, I find that in markets in which innovation is by nature frequent, some degree of patent breadth encourages innovation (Result 9).

In Section 5, I study optimal patent policy. To do so, I assume that firms participate in an economy in which innovations are described by a quality ladder under which each innovation is regarded as a quality improvement over the previous one. Assuming a representative consumer and competition in the entrants' market, I study how optimal patent breadth and length varies with market characteristics, such us the natural innovation rate, the magnitude of the quality improvement, and whether the planner has to provide incentives to create the market or if the market already exists. I find that the optimal patent length is decreasing in both the natural innovation rate and the average quality improvement, regardless of the existence of the market. Similarly, the optimal patent breadth is also decreasing in the natural innovation rate and the average quality improvement. When no market exists, larger patent breadth does not deter potential entrants from innovating, as there is no first patent to infringe. On the contrary, breadth increases the value of becoming the first incumbent in the market. This creates a trade-off between incentivizing the market's creation and discouraging posterior improvements. This trade-off generally leads to a large breadth, which decreases in markets that innovate faster, where the prospects of future innovation are more important.

When the planner is given the opportunity to revise his policy, I find that stronger policies are adopted. The high level of breadth chosen before the market's creation inhibits the entrant's innovation in future races. Since the only tool that the planner possesses to provide incentives for innovation is patent policy, stronger policies are adopted in order to increase the reward of the firms that compete at the moment of the policy redesign. This crates a self-reinforcement effect that generates stronger policies.

Section 6 presents different extensions of the model. Section 6.1 extends the model to the case in which license fees are paid in the form of royalties. In this scenario, the *entrant's* incentives for innovation are also non-stationary and increasing through time. This is driven by the fact that if an entrant innovates, he may infringe the active patent and be forced to pay license fees. If these fees are paid only while an infringement occurs (a royalty), the expected license fees decrease as the expiration date of the active patent approaches, increasing the entrants' value for an innovation. Section 6.2 studies the case in which an incumbent may reach a lead of n technological steps above his competitors. I show that the finding that an extension of patent protection may decrease the incumbent's innovation rate is robust to this specification.

Finally, due to the large previous literature, I devote Section 7 to discuss how my results relate to the previous work in patents and Section 8 concludes.

2 A Simple Model of Innovation

Set up Consider a continuous time economy consisting of firms competing in an infinitely-lived market with the opportunity to perform R&D to gain temporary competitive advantages through the development of higher quality goods. At each instant of time t, there is at most one firm possessing an active patent for the latest technology available in the market; this firm is called the *incumbent* and is denoted by i. All other firms compete in the market using the second-best available technology and are called *entrants*, denoted by n, as they perform R&D in order to leapfrog the incumbent and become the new technology leader. I assume that the incumbent is a long-run player, who maximizes the present discounted value of his profits, whereas the entrants are an infinite sequence of short-run investors that maximize their instantaneous payoffs.[1] The incumbent discounts his future payoffs at a rate $r > 0$.

The value of an innovation depends on the underlying patent system, the monopoly profits that the incumbent gets while he holds the patent, and the time remaining until the patent's expiration date. A *patent system* consists of a *statutory length* $T \in \mathbb{R}^+$ and an *infringement probability* $b \in [0, 1]$. I assume that all innovations are patentable but innovations may infringe the previously active patent with probability b. While an incumbent's infringement of its own patent is irrelevant, entrants have to pay a compulsory license fee. The license fee is a fixed proportion $\alpha \in [0, 1]$ of the value of the patent v_o, to be determined in equilibrium.[2] For all the participants of this economy the tuple (T, α, b) is considered common knowledge and exogenously given.

While the patent is active, the incumbent receives a monopoly flow of profits $\pi > 0$. When a patent expires, the incumbent's competition is legally able to use the latest technology, driving the incumbent's profits to zero. Finally, when an innovation occurs, the non-succeeding firms are able to invent around the second-best technology driving the profits of all non-incumbents (who do not have the leading technology) to zero. In concrete terms, this assumption implies that there is, at most, a one-step lead between the technology leader and his competitors.[3]

A *period* is the time lapse between two innovations. Periods have random length and

[1] Alternatively, the model can be re-interpreted as one with a continuum of entrants; see footnote 4

[2] The assumption that the compulsory license fee is a fixed fraction of value of a new patent, and not a fraction of the profits while the infringement lasts, is for the sake of obtaining an analytical solution. This alternative assumption is analyzed in Section 6.1.

[3] In section 6.2 I analyze the extension under which, in the context of a duopoly, the technology leader can be n-steps above his competitor. As shown there, the main force driving the results is still present there.

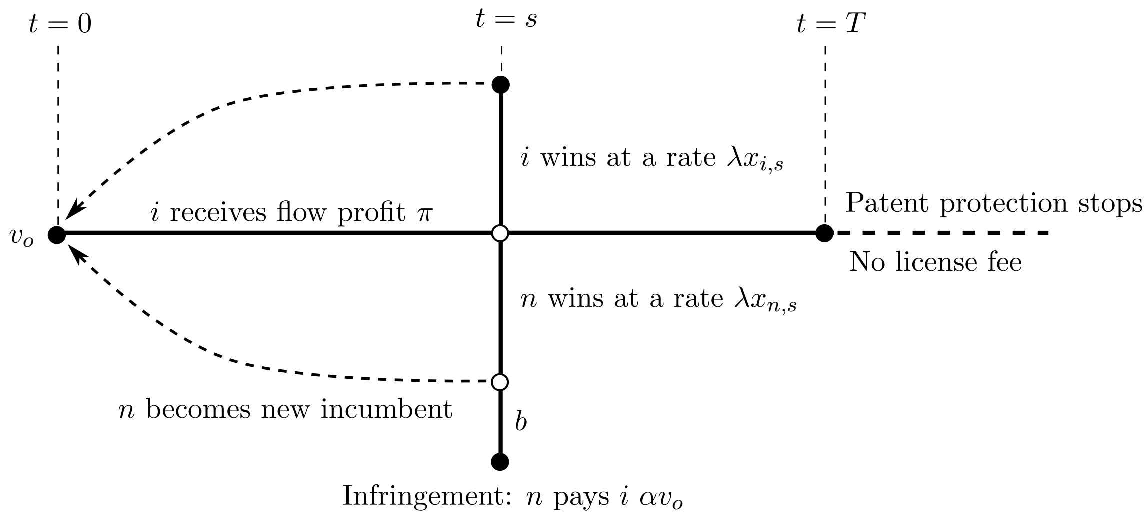

FIGURE 1: Timing of the game

are basis of the notation below; $t = 0$ represents the beginning of a period and $t = s$ represents that s units of time have passed since the last innovation; that is, the current active patent has $T - s$ time left.

In order to obtain an innovation firms have to invest in R&D. These investments, also called R&D efforts, lead to a stochastic arrival of inventions which is an increasing function of the firms' investments. At every t, each firm $k \in \{i, n\}$ chooses an R&D effort $x_{k,t} \geq 0$. The instantaneous cost of this effort is given by the cost function $c(x_{k,t})$. I assume that the cost is a twice continuously differentiable function that satisfies $c'(\cdot)$, $c''(\cdot) > 0$, $c'(0) = 0$ and $\lim_{u \to \infty} c'(u) = \infty$ (in order to obtain an analytical solution below I will focus on the case in which $c(x_{k,t}) = (x_{k,t})^2/2$.)

Firm's k effort induces an arrival of innovations described by a non-homogeneous Poisson process whose arrival rate at instant t is $\lambda x_{k,t}$ with $\lambda > 0$. The parameter λ is called the market's *natural innovation rate* and is a measure of the pace of innovation in any given market. The Poisson processes are independent among firms and generate a stochastic process that is memoryless but potentially non-stationary. The waiting time between two innovations is described by an exponential distribution with the probability of observing a success by instant t equal to $1 - \exp(-z_{0,t})$ where $z_{\tau,t} = \lambda \int_\tau^t (x_{i,s} + x_{n,s})\, ds$ is a measure of the accumulated effort from instant τ to instant t.

Timing of the game The timing of the game, depicted in Figure 1, is as follows. When an innovation arrives our time index t is reset to zero. From that time and on, and while his incumbency lasts, the patent holder receives a monopoly flow of π whereas the entrants obtain zero profit from the competitive market they are in. At each instant of time, the incumbent faces a different entrant. Entrants are assumed to play only once in the game maximizing their instantaneous payoff. Thus, the effort $x_{n,t}$ represents how the investment of the different entrants evolves through time.[4] At every instant of time, both incumbent and entrant choose their effort simultaneously determining the arrival rate of innovation for both firms. When an innovation occurs, the succeeding firm becomes the new incumbent and competitors are able to invent around the now second-best patent keeping the technological gap to a one-step lead. If the innovating player is an entrant, with exogenous probability b his innovation is considered to infringe the incumbent's patent and he has to pay a compulsory license fee equal to a proportion α of the expected value of the project v_o which is determined in equilibrium. If no innovation has occurred within the statutory length of the patent, the patent holder loses his incumbency status and becomes one of the many entrants of the game. Consequently, no license fee can be charged for innovations that occur after T.

Payoffs and strategies Fix $\{x_{n,t}\}_0^T$. From the perspective of time s , the incumbent's value of having a patent that has been active for s years is:

$$v(s) = \max_{\{x_{i,t}\}_s^T} \int_s^T \left(\pi + \lambda x_{i,t} v_o + \lambda x_{n,t} b \alpha v_o - c(x_{i,t})\right) e^{-z_{s,t}} e^{-r(t-s)} dt. \tag{1}$$

That is, with probability $\exp(-z_{s,t})$ no innovation has occurred between instant s and t and the patent is still active at t. At that instant t, the incumbent receives the flow payoff π and pays the cost of his effort $c(x_{i,t})$. The effort leads him to an innovation at a rate $\lambda x_{i,t}$ obtaining the benefit of a brand new patent v_o. On the other hand, the entrant may succeed at a rate $\lambda x_{n,t}$ in which case he may infringe the current patent with probability b, having to pay to the incumbent a compulsory license fee of αv_o. All of these payoffs are discounted by $e^{-r(t-s)}$.

On the other hand, at each instant of time t at which a patent is active, a new entrant decides how much to invest. Entrants maximize their instantaneous flow payoff

$$\lambda x_{n,t} \left(b(1-\alpha) v_o + (1-b) v_o\right) - c(x_{n,t}).$$

This is, the entrant's expected profit from an innovation $(1-b\alpha) v_o$, adjusted by the arrival rate induced by his effort $\lambda x_{n,t}$, minus the cost of his effort $c(x_{n,t})$. It is immediate that

[4] This assumption can be thought of a reduced form of the following model: There are two arrival processes, one for the incumbent and one for the entrants. All the potential entrants play simultaneously. The arrival rate of the entrants depends on the sum of their efforts denoted by $x_{n,t}$. If the entrants' arrival process delivers an innovation, the succeeding firm is chosen with an uniform probability among all possible entrants. When the number of entrants is arbitrarily large, each firm has efforts and payoffs that converge to zero but on the aggregate total effort, $x_{n,t}$, does not.

each entrant will choose the same effort through time, which is given by

$$x_n = c'^{-1} \left(\lambda \left(1 - b\alpha \right) v_o \right). \tag{2}$$

Similarly, when no patent is active, and no license fee can be charged for an innovation, the entrant's effort jumps to $x_n = c'^{-1} \left(\lambda v_o \right)$.

In this game a strategy for the incumbent is a mapping from the active life of his patent t to an effort level. Therefore, this is a Markov game as both the incumbent's and entrants' strategies are functions of this state variable only. This work focuses on finding Markov Perfect equilibria.

3 The Incumbent's Problem

This section solves the incumbent's problem using optimal control techniques. I start by applying the Principle of Optimality to derive the Hamilton-Jacobi-Bellman (HJB) equation which provides necessary and sufficient conditions for a maximum. Maximizing the HJB equation we will find the optimal effort rule which is used to solve for the value of having a patent at each t. In the previous steps the value of a new patent v_o is taken as given; the section concludes by showing that there exists a value of v_o compatible with the solution found above.

Observing equations (1) and (2) it can be appreciated that α and b are indistinguishable in their effects; therefore, for simplicity I use $\beta = b\alpha \in [0, 1]$ for the combined effect of the license fee α and the infringement probability b. In other words, β represents a measure of the patent breadth.

Starting at an arbitrarily small time interval $[t, t + dt)$, the incumbent's value of having a patent for t years must satisfy the Principle of Optimality:

$$v\left(t\right) = \max_{x_{i,t}} \left\{ \left(\pi + \lambda x_{i,t} v_o + \lambda x_n \beta v_o - c\left(x_{i,t}\right)\right) dt + e^{-rdt} \mathbb{E}\left[v\left(t + dt\right) | x_{i,t}, x_n \right] \right\}.$$

That is, given the optimal strategy, the value of having a patent at t is equal to the payoff flow at that instant of time, plus the discounted expected continuation value of the patent. When dt is small enough, the discount factor $\exp\left(-rdt\right)$ is equal to $1 - rdt$. On the other hand, the expected continuation value of the patent $\mathbb{E}\left[v\left(t + dt\right) | x_{i,t}, x_n\right]$ is equal to the probability of not having an innovation today $1 - \lambda\left(x_{i,t} + x_{n,t}\right) dt$, times the value of a patent tomorrow $v\left(t + dt\right) = v\left(t\right) + v'\left(t\right) dt$, plus the probability that an innovation occurs $\lambda(x_{i,t} + x_n) dt$ times the continuation value of the *current* patent after an innovation which is zero. Substituting back the previous expressions in the equation derived from the Principle of Optimality, and ignoring all the terms that are of order $(dt)^2$, I obtain the incumbent's HJB equation

$$0 = \max_{x_{i,t}} \left\{ \pi + \lambda x_{i,t} \left(v_o - v\left(t\right) \right) + \lambda x_n \left(\beta v_o - v\left(t\right) \right) - c\left(x_{i,t}\right) + v'\left(t\right) - v\left(t\right) r \right\}. \tag{3}$$

Condition (3) is necessary and sufficient for a maximum, and the optimal effort rule is determined by its first-order condition

$$c'\left(x_{i,t}\right) = \lambda \left(v_o - v\left(t\right) \right). \tag{4}$$

Equation (4) is very informative about the incumbent's effort dynamics. It states that, at any instant t, the marginal benefit of effort is equal to the *incremental* rent that the incumbent obtains from innovating. That is, the expected profits from a new patent v_o, minus the expected profits loss from giving up the currently active patent $v(t)$. Since the value of having a patent declines with the proximity of its expiration date (see Proposition 1), the incumbent's effort increases through time. The incumbent starts performing zero effort at $t = 0$ as, by definition, $v\left(0\right) = v_o$. Effort reaches its maximum at $t = T$, where the replacement value of the currently active patent becomes zero, and the incumbent invest at a rate of λv_o. The intuition of this result is as follows: when t years have passed since the incumbent's innovation, his benefit from a new patent is just to extend his patent protection for another t years. Therefore, when the incumbent just obtained a new patent he obtains no benefit from renewing it. The benefit increases with time and becomes maximal at the end of the statutory life, when a patent offers no protection.

To be able to obtain an analytical solution, more structure for the cost function is needed. I assume $c\left(x_{k,t}\right) = \left(x_{k,t}\right)^2/2$. Substituting the incumbent's and entrant's effort (equations (4) and (2) respectively) into (3), using the cost assumption, and rearranging, the following ODE is obtained

$$0 = v'\left(t\right) + \frac{\lambda^2}{2}\left(v\left(t\right)\right)^2 - \left(r + \lambda^2\left(2-\beta\right)v_0\right)v\left(t\right) + \pi + \lambda^2\left(\frac{1}{2} + \left(1-\beta\right)\beta\right)v_o^2. \tag{5}$$

This is a separable Riccati differential equation. It has a unique solution that satisfies the boundary condition $v(T) = 0$ which is given by (see Appendix A for the details)

$$v\left(t, v_o\right) = \frac{1}{\lambda^2}\left(\frac{\left(\theta^2 - \phi^2\right)\left(e^{\phi(T-t)} - 1\right)}{\left(\theta + \phi\right)e^{\phi(T-t)} - \left(\theta - \phi\right)}\right) \tag{6}$$

where

$$\theta = r + \lambda^2\left(2-\beta\right)v_o \text{ and } \phi = \sqrt{\theta^2 - \lambda^2\left(2\pi + \lambda^2\left(1 + 2\left(1-\beta\right)\beta\right)v_o^2\right)}.$$

The solution to the incumbent's problem is, however, not complete as we have solved the problem as a function of the value of a new patent v_o. In order to have a solution it is necessary to show that a fixed point to $v(0, v_o) = v_o$ exists. It is important to note here, that the function $v(t, v_o)$ is well-defined for all non-negative values of v_o. In particular, this is also true for values of the parameters such that ϕ is imaginary because all the imaginary terms in (6) cancel out (see Appendix A.) The next proposition establishes existence of the fixed point $v(0, v_o) = v_o$.

Proposition 1 (Existence) *There always exists a fixed point $v(0, v_o) = v_o$. The value of having a patent at t is given by equation (6) evaluated at v_o. The incumbent's R&D investments are given by the continuously increasing function of t*

$$x_{i,t} = \frac{1}{\lambda}\left(\frac{(\phi-\mu)(\theta+\phi)e^{\phi(T-t)} + (\theta-\phi)(\phi+\mu)}{(\theta+\phi)e^{\phi(T-t)} - (\theta-\phi)}\right) \tag{7}$$

where $\mu = r + \lambda^2(1-\beta)v_o$.

The proof of Proposition 1 is relegated to Appendix B.1. Unfortunately, uniqueness of the fixed point cannot be guaranteed for all sets of parameters. However, since there is a finite number of fixed points and they can be ranked in a Pareto sense, i.e. all players rank the fixed points in terms of their payoffs equally. In the subsequent analysis, I use the Pareto-dominant fixed point as the solution, i.e. the largest v_o that is a fixed point.

4 R&D Dynamics

The purpose of this section is to understand how the different elements of this economy affect the firms' R&D investments. Due to the complexity of the problem, a combination of theoretical analysis and computational techniques are used to characterize how the R&D dynamics react to changes in the economy.[5]

From the previous section we know that entrants perform constant investments equal to $\lambda(1-\beta)v_o$ while the patent is active, and jump to λv_o when patent protection is over. On the other hand, at the beginning of the patent protection, the incumbent's investments start at zero and slowly increase to λv_o at the patent expiration date.

The next set of propositions and results explain how different elements of the model affect the value of a new patent.

Proposition 2 (Value of a new patent) *The value of the patent v_o increases with:*

[5] To be precise about the difference between a proven result and a numeric one, I use the term "Proposition" to state the former and "Result" for the latter.

1. *an increase of the statutory length T;*

2. *a decrease the interest rate r;*

3. *an increase in the profit flow π;*

The claims in Proposition 2 are intuitive. If patent protection lasts longer or if the discounted flow of monopoly profits is higher, the value of a new patent must go up.

Result 3 (Value of a new patent and breadth) *An increase in patent breadth increases the value of a new patent, v_o, except for values of β that are close to one. In which case, v_o may decrease with an increase in β.*

Holding the value of a new patent fixed, there are two direct effects interacting with changes in β.[6] First, an increase in β decreases the entrants' innovation rate $\lambda x_n = \lambda^2(1-\beta)v_o$. This decrease in λx_n can be understood as a decrease in the incumbent's competition and, consequently, as an increase in the expected patent duration. Secondly, in the event that an entrant succeeds, an increase in breadth increases the expected payoff that the incumbent receives, βv_o. This increase in the incumbent's expected payoff is due to the interaction of two effects: the increase in the probability that an entrant's success is considered to infringe the incumbent's patent, and an increase in the license fee paid in such case.

Even though the effects of an increase in competition and the increase in revenue derived from an entrant's success, taken independently, push the value of a new patent up, the interaction of the two may create a countervailing effect. The rent that an incumbent perceives from patent breadth is given by

$$\lambda x_n \beta v_o = \lambda^2 \beta(1-\beta)(v_o)^2. \tag{8}$$

Therefore, while larger breadth increases the expected payoff obtained from an entrant's success, it also decreases the rate at which entrants are succeeding. Thus, as seen in equation (8), the expected payoff derived from patent breadth increases with β for values of $\beta \leq 1/2$ and decreases otherwise. However, the decrease in the expected rent obtained from patent breadth may only overcome the effect of an ease in competition in cases where β is very close to 1, i.e. where the marginal gain from less competition is almost zero, but the effect of higher expected revenue is highest. Result 4 is depicted in Figure 2a. Figure 2a

[6]There is, of course, an indirect effect through changes in the firm's investments due to the change in v_o. These effects cannot change the direction of the result, they can only potentiate or diminish the magnitude of the direct effects.

depicts Result 3 for different values of the natural innovation rate λ. As can be observed, breadth generally increases the value of a new patent, except for values of β close to 1. Also, Figure 2a shows that higher values of λ increase the effect of the revenue loss due to the increase in patent breadth. This last effect can be seen directly from equation (8), which is increasing in λ.

Result 4 (Value of a new patent and natural innovation rate) *An increase in the natural innovation rate λ:*

1. *increases the value of a new patent, for values of β close to 1,*
2. *decreases the value of a new patent, for values of β close to 0.*

For clarity, I explain the direct effect of changes in the natural innovation rate λ without considering indirect effects that changes in v_o may bring. An increase in the natural innovation rate λ has two opposing effects. On one hand, the increase in λ pushes the value of a new patent down as it increases the arrival rate of the incumbent's competition λx_n. On the other hand, the increase in λ pushes the value of a new patent up as it increases the effectiveness of the incumbent's effort, reducing the cost of producing a new innovation. The net effect depends on the relative magnitude of the incumbent's and entrants' investments. Therefore, when β is close to zero, entrants perform higher effort rates relative to that of the incumbent, decreasing the value of a patent. Similarly, when β is close to one, the incumbent faces little competition and the cost-saving effect dominates, increasing the value of a new patent.[7] Result 4 is depicted in Figure 2a, where it can be appreciated how the effect of an increase in λ varies with patent breadth.

Proposition 5 (Patent length and effort) *An increase in the statutory length T:*

1. *increases the entrant's effort;*
2. *if the increase in length does not apply to the current active patent, increases the incumbent's effort at each t;*
3. *if the increase in length applies to the current active patent, decreases the incumbent's effort at each t.*

[7] In the case of $\beta = 1$ it is possible to give a formal proof. When $\beta = 1$, the entrants' effort are zero and, consequently, there is no increase in competition due to an increase in λ. Therefore, the increase in λ is cost saving only, increasing the value of v_o.

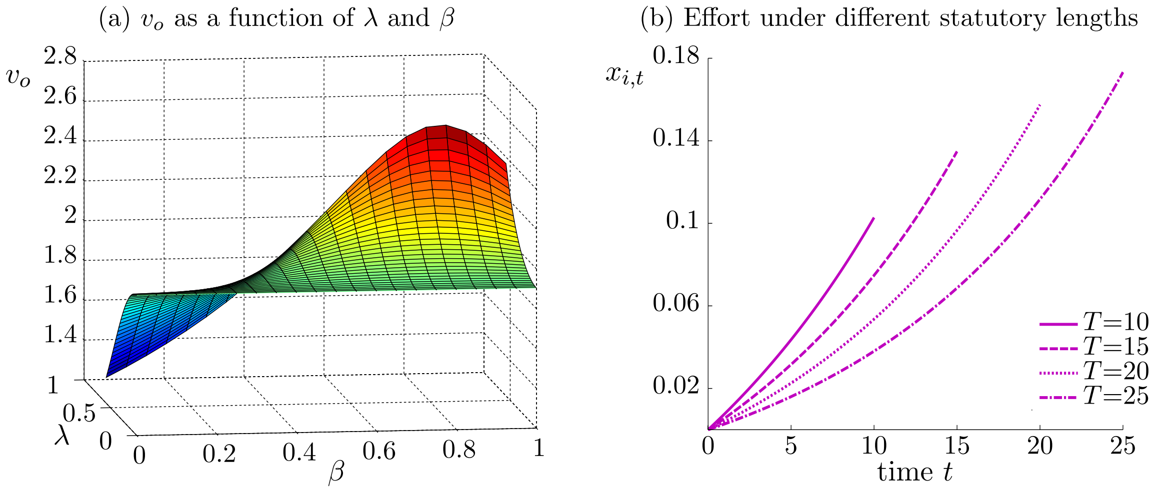

FIGURE 2: Comparative Statics.

The prize that a firm gets from innovating at time t is given by incremental rent at that instant in time. As shown in Proposition 2, an increase in the statutory length of a patent leads to an increase of the value of obtaining a new patent v_o. Consequently, the increase in v_o increases the incremental rent of an entrant, increasing the entrants' effort rate (Proposition 5.1). For the incumbent, on the other hand, when a patent extension only affects future innovations, his prize for innovating $(v_o - v(t))$ is affected directly through the increase in v_o, and indirectly through the effect that a change in v_o has on investments, and hence the continuation value of the current patent $v(t)$. Proposition 5.2 states that the net effect is such that the incumbent's incremental rent increases, increasing his effort rate at t.

A change in the patent length that applies to the current patent has, however, a direct effect on the value of the active patent $v(t)$. In particular, $v(t)$ increases with an extension of patent protection. Moreover, the increase in $v(t)$ is greater than the increase in v_o, reducing the incumbent's prize obtained from an innovation $v_o - v(t)$ and therefore decreasing the incumbent's effort rate at t. This result is driven by the fact that the *effective* length of a patent generally differs with its *statutory* length. Then, when offered longer patent protection, the increase in the value of an active patent at $t > 0$ is greater than the increase at $t = 0$, because the probability of actually reaching and using the patent extension is higher the closer the incumbent is to the expiration date T. This is captured by Proposition 5.3 and depicted in Figure 2b where I show the incumbent's effort through time for different statutory lengths. There, it can also be observed that, even though the innovation rate of the incumbent decreases at each t, the average effort performed by the incumbent

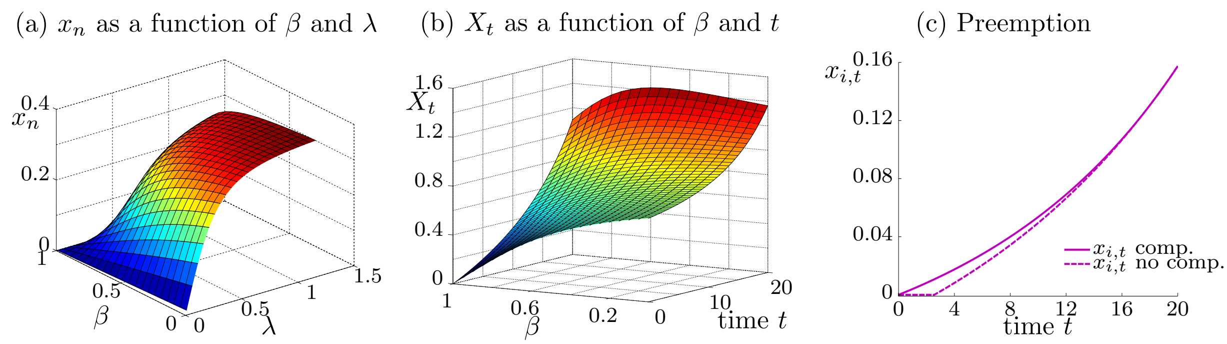

FIGURE 3: Comparative Statics 2

throughout the patent life may go up because patents last longer and the terminal effort is higher. At the end of this section I explore how an increase in patent length affects the economy's innovation rate.

When the analysis of patent length is restricted to the context of a single-innovation model, the only effect that can be captured is the increase in the value of a new patent v_o. This restriction leads to the conclusion that longer patent protection always conveys higher innovation rates. By incorporating a sequential structure to the innovation process, I find that the replacement value of an active patent depends on patent policy as well. As a consequence, longer patents decrease the incumbent's innovation rate and, as we shall see below, may decrease the economy's innovation rate.

Result 6 (Patent breadth and effort) *An increase in the patent breadth β:*

1. *decreases the entrants' effort x_n,*

2. *increases the incumbent's effort $x_{i,t}$ at each t whenever v_o increases with β.*

To understand the content of Result 6.1, recall that the entrants' effort at any t is given by $\lambda(1-\beta)v_o$. As mentioned above, the direct effect of an increase in β is to decrease x_n. Result 6.1 states that the direct effect always dominates any indirect effect that an increase in β may induce over v_o. This is shown in Figure 3a, where I depict the entrants' effort rate for different values of β and λ.

When an increase in patent breadth increases the value of a new patent, it can be shown that it is also increases the value of $v(t)$ for all $t \leq T$. Result 6.2 states that the increase in the value of a new patent always dominates the increase of $v(t)$.

The effect of an increase in patent breadth on the total effort $X_t \equiv x_{i,t} + x_n$ is not clear cut. At $t = 0$, total effort decreases as the entrant's effort goes down and the incumbent performs no effort. At $t = T$, when $x_{i,t} = \lambda v_o$, the increase in the incumbent's effort may well overcome the decrease in x_n, increasing total effort. Figure 3b shows the evolution of total effort through time for different values of β. At $t = 0$, effort decreases with larger breadth as the reduction in the entrant's effort is the only active effect. As t approaches the patent expiration date T, we can observe the increase in the incumbent's effort rate. For larger values of β and while the increase in β increases the value of a new patent, we observe that total effort may increase due to an increase with β. When β is close to one, as seen in Result 3, the value of a new patent decreases, and so does X_t. The net effect of breadth in the economy's arrival rate is discussed at the end of this section.

The combination of Proposition 5 and Result 6 give clear empirical predictions about what we should observe in different markets. First, we are more likely to observe an incumbent innovating near the end of his patent protection. Secondly, the rate of innovation for the entrant is strictly related to the perceived extent of the patent breadth. If innovations are likely to be considered an infringement, or if the institutional background (like courts) tend to be favorable with patent holders, entrants are less likely to innovate during the period of the patent protection. On the other hand, if infringements are hard to determine we should observe more frequent and a higher rate of innovation coming from potential entrants.

The next two propositions relate the findings in my model with some of the results in the literature.

Proposition 7 (Stationarity) *Under an infinite statutory patent length, the incumbent investments are constant and equal to zero. The value of a patent is independent of t and equal to*

$$v_\infty = \frac{1}{2\lambda^2 (1-\beta)^2} \left(-r + \sqrt{r^2 + 4\pi\lambda^2 (1-\beta)^2} \right). \tag{9}$$

Proposition 7 shows that when patent protection is infinitely long, incentives become stationary and incumbents have no incentives to innovate. This is due to the fact that a new innovation just replaces the current active patent with one of the same value. The intuition of this result is that, under infinite patent protection, the benefit of innovating does not change as time passes; at any point of the patent life the incumbent is protected for an infinite amount of time. Proposition 7 shows the connection that exists between my model and those that assume infinite patent protection in the context of sequential innovation. In these papers, incentives are stationary and it is typically assumed that R&D is performed only by potential entrants. Proposition 7 is, in some sense, a justification for the assumption of passive incumbents (although my framework is not the first justification proposed, see for example Aghion and Howitt (1992).) The next result captures how the existence of competition affects the incumbent's investment decisions.

Result 8 (Preemption) *If the incumbent were to face no competition ($x_n = 0$):*

1. *fixing the value of the project v_o, the incumbent performs less effort at each $t < T$.*

2. *depending of the initial value of β, the value of a new patent may increase or decrease.*

3. *if v_o goes up, so does the incumbent effort for all values of t.*

Some authors have argued that patent-supported monopolies are likely to persist as incumbents may have more incentives to invest than entrants as the former may have more to lose than the latter to gain. This effect is referred in the literature as the *preemption effect* (see Gilbert and Newbery (1982) for an example.) As shown in Figure 3c, Result 8 states that the model contains preemption incentives. If we hold the value of a new patent fixed, the lack of competition decreases the incumbent's incentives to perform R&D.

However, the sequential nature of the game makes the lack of competition affect the value of a new patent. Since the restriction $x_n = 0$ is equivalent to impose $\beta = 1$, whether the value of a new patent increases or decreases will depend on the initial value of β from which the comparison is made. Hence, as in Result 6.2, the incumbent's effort will increase whenever the value of a new patent goes up.

To conclude this section, I compare different patent policies in terms of their capacity to generate higher innovation rates. In many games of experimentation (see Reinganum (1982) or more recently Keller *et al.* (2005)) the innovation rate is approximated by the total investment performed during the game. This methodology, however, is not an appropriate measure for a game of infinite length as investments never stop. Another option is to restrict attention to the total investments performed during patent protection $z_{0,T}$, though this ignores the R&D performed once patent protection is over which is also affected by patent policy. Using the property that successes follow a non-homogeneous exponential distribution, I define the economy's arrival rate to be the inverse of [8]

$$\mathbb{E}[t] = \int_0^\infty t\lambda(X_t)e^{-z_{0,t}}dt,$$

as the measure for comparison among different patent policies. [9]

[8]In the case of constant total effort equal to one, the distribution of successes will follow an exponential distribution with an arrival rate equal to λ. In that case, since $\mathbb{E}[t] = 1/\lambda$, the measure $1/\mathbb{E}[t]$ corresponds exactly to the arrival rate of the economy.

[9]Other authors like Bonatti and Hörner (2011) used stochastic dominance (SD) to compare the distribution of successes for different sets of parameters. Here, the lack of monotonicity in the comparison of policies makes SD uninformative.

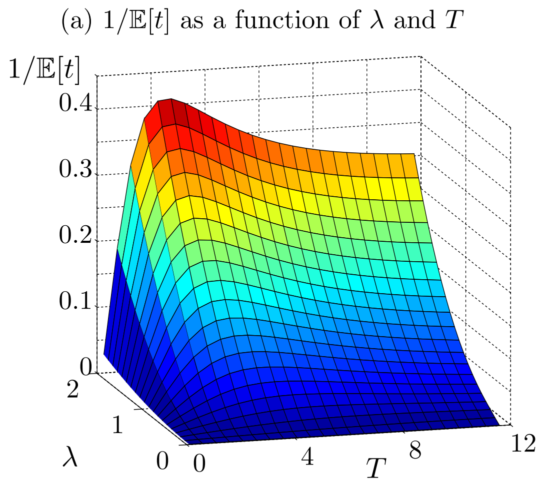

FIGURE 4: The economy's innovation rate

Result 9 (The economy's innovation rate) *The economy's innovation rate,* $1/\mathbb{E}[t]$*, is maximized at a finite patent length* T*. The breadth that maximizes* $1/\mathbb{E}[t]$ *is zero for low values of the natural innovation rate* λ *and increases with it.*

Figure 4 depicts the economy's arrival rate, $1/\mathbb{E}[t]$, under different set of policies and parameters. Figure 4a shows how $1/\mathbb{E}[t]$ varies with patent length and the natural innovation rate, fixing $\beta = 0$. We can observe that the effect of patent length is non-monotonic. On one hand, low values of T do not give enough rewards to induce high rates of innovation. On the other hand, when patent protection is too long, the decrease in the incumbent's innovation rate becomes a first-order effect whereas the increase in the entrants' effort slows down as it converges to λv_∞.

Figure 4b depicts how $1/\mathbb{E}[t]$ varies with patent breath and the market's natural innovation rate fixing $T = 10$. For low values of λ, the optimal patent breadth is zero and it increases as the natural innovation rate increases. The intuition for this result is as follows, when the natural arrival rate is high, innovation occurs at a higher frequency shortening the patent's effective life. In that case, patent length is an ineffective tool to encourage innovation and breadth becomes the more effective tool to promote innovation.

5 Quality Ladder and Patent Policy

In this section I study how patent policy affects the social welfare of an economy and, in particular, I study how optimal patent policy varies with the market's natural innovation rate λ.

The framework developed in the previous sections is insufficient to address this question as the introduction of consumers is necessary to assess both the value of an innovation and the cost of inducing it. Consequently, I add more structure to the problem by assuming that the economy is characterized by a quality ladder in which new products are regarded as quality improvements on the previous ones. Consumers value higher quality goods more than those of lower quality. Therefore, since innovation is, up to some point, promoted through the grant of monopoly rights, there exists a trade-off between the benefit of future innovations and the benefit of competitive markets.

5.1 Consumers and Firms

I assume an economy characterized by a representative consumer that regards the goods as perfect substitutes from one another. In particular, I assume the following preferences

$$u_{j,t}(\mathbf{q_j}) = \sum_{m=1}^{j} \gamma^{m-1} q_m \tag{10}$$

where j corresponds to the number of breakthroughs that have occurred in the market, $\mathbf{q_j}$ is j-dimensional vector representing the consumption of each of the j qualities available in the market, and $\gamma > 1$ is the fixed increase in quality induced by an innovation. In simple words, under these preferences, consumers value a unit of the newest quality γ times more than a unit of the second-best quality.

At each point in time the consumer's income is normalized to 1. This income is completely expended as I assume that there is no financial market to smooth consumption out. These assumptions imply that the consumer's demand will be time independent in the sense that the number of breakthroughs does not alter the quantity demanded of the highest-quality good, it only alters the identity of the supplier.

Firms are assumed to compete à la Bertrand and to have a marginal cost of production $c > 0$ for all possible qualities. Since the maximal technological gap in our economy is one, the second-best (and any older technology) will be supplied competitively and at a price of c. While patent protection lasts, profit maximization leads the incumbent to charge a price $p_i = c\gamma$ to sell $q_i = 1/c\gamma$ units of his good and to obtain profits equal to $\pi = q_i(p_i - c) \equiv (\gamma - 1)/\gamma$.[10] Also, the highest quality good is the only good traded in the

[10] To see this, observe that if consumers where to spend all their income in the second-best technology, they would obtain a quantity of 1/c and a utility of γ^{j-2}/c. Then, the incumbent charges the price that makes consumers indifferent between her technology and the second-best technology. That is, the incumbent chooses a price such that if the consumers were to spend all their money in the highest quality good, they get the same utility as they would obtain expending their money in the low quality good, i.e.

market and the consumer's utility is equal to $u_j^P = \gamma^{j-2}/c$ where P denotes that patent protection is active.

When patent protection expires, perfect competition drives the price of the high quality good to the marginal cost of production. The consumed quantity is $q_i = 1/c$, and the consumers' utility becomes $u_j^C = \gamma^{j-1}/c$ where C denotes a competitive market. The ratio $u_j^C/u_j^P = \gamma$ is independent of the number of innovations, meaning that the cost, in terms of consumer surplus, of granting a patent is stationary and independent of the number of breakthroughs in the market.

5.2 Total Surplus

The total surplus consists of the expected sum (integral) of the discounted utility flow, plus the incumbent's profits, minus the costs of R&D at each instant of time. The previous expectation is taken over all possible paths that innovation may generate. Although the problem is quite complex due to the non-stationarity of efforts and of the stochastic process, a well-defined expression for the total surplus is found using the periodic structure of the game. For simplicity, I separate the computations of the consumer and producer surplus. Under previous definitions $u_{j,t}$ (equation (10)) is equal to $u_{j,t} = u_j^P$ if $t < T$ and to $u_{j,t} = u_j^C$ otherwise. Then, the consumer's surplus at the moment that the j-th innovation has occurred is given by

$$CS_j = \int_0^\infty \left(u_{j,t} + \lambda X_t CS_{j+1}\right) e^{-z_{0,t}} e^{-rt} dt \tag{11}$$

That is, with probability $\exp(-z_{0,t})$, no innovation has arrived by time t; at that time, consumers receive the utility flow $u_{j,t}$ and, at a rate of λX_t, a new innovation arrives and consumers get the expected present value of the utility flow after $j+1$ innovations. Let S_t be any measurable function of time; the operator $\mathbb{DE}[S_t]$ corresponds to the expected discounted value of S_t, i.e.

$$\mathbb{DE}[S_t] = \int_0^\infty S_t e^{-z_{0,t}} e^{-rt} dt.$$

Since $u_{j+1,t} = \gamma u_{j,t}$ for all t, we observe that it must be the case that $CS_{j+1} = \gamma CS_j$ and the following expression for CS_1 is obtained

$$CS_1 = \frac{\mathbb{DE}[u_{1,t}]}{1 - \gamma\lambda \mathbb{DE}[X_t]}. \tag{12}$$

The consumer surplus after the first innovation corresponds to the present value of a growing perpetuity. This perpetuity is given by the ratio of the expected discounted utility

$\gamma^{j-1} q_i = j^{j-2}/c$ which implies $q_i = 1/c\gamma$, therefore $p_i = c\gamma$.

flow $\mathbb{DE}[u_{1,t}]$ and the interest rate adjusted by growth $1 - \gamma\lambda\mathbb{DE}[X_t]$.[11]

Let π_t be equal to π for $t \leq T$ and to zero otherwise. Following a similar procedure, I find that the producer surplus, at the beginning of any period, is equal to

$$PS = \frac{\mathbb{DE}[\pi_t - x_{i,t}^2/2 - x_{n,t}^2/2]}{1 - \lambda\mathbb{DE}[X_t]}. \tag{13}$$

Observe that the denominator of (13) differs from that in (12) as the producer surplus does not grow at a rate of γ when a new innovation arrives.

5.3 Patent Policy: New and Mature Markets

In this section I study how patent policy affects the total welfare of an economy. In particular, I am interested in investigating how the optimal policy varies with different market characteristics. In doing so, I compare the incentives that a planner would provide in environments where the market does not exists, to those incentives that would be provided in an environment where the first product has already been developed.

In an environment where no market exists, a situation which I call a "new market," the planner implements

$$(T, \beta) \in \underset{(T,\beta)\in[0,\infty)\times[0,1]}{\operatorname{argmax}} W_{new} \equiv \int_0^\infty (\lambda^2 v_o TS - \frac{(\lambda v_o)^2}{2})e^{-(\lambda^2 v_o + r)t}dt \tag{14}$$

where $TS = CS_1 + PS$. That is, when no innovation has yet occurred, we have a sequence of entrants performing a constant effort $x_n = \lambda v_o$ until the first innovation arrives. This effort is performed at a cost of $(\lambda v_o)^2/2$ per instant of time. Innovations arrive at a rate $\lambda^2 v_o$, and when an innovation occurs, the expected total surplus TS is obtained. It is important to note here that TS is also a function of the policy variables (T, β) which makes the problem hard to solve analytically. I denote the solution of problem (14) by $p_0 \equiv (T_o, \beta_o)$.

To understand the planner's incentives after the market is created, a situation which I call a "mature market," I study the solution to:

$$\max_{p\in[0,\infty)\times[0,1]} W_{mat} \equiv \int_0^\infty (\lambda X_t(p_0)TS(p) - \frac{(x_{i,t}(p_0))^2}{2} - \frac{(x_n(p_0))^2}{2})e^{-z_{0,t}(p_0)}e^{rt}dt. \tag{15}$$

[11] For the sake of intuition, equation (12), in a case where effort is constant and equal to 1, becomes

$$CS = \frac{u_p + (u_c - u_p)e^{(\lambda+r)T}}{r - \lambda(\gamma - 1)},$$

which corresponds to the expected discounted utility that grows at a rate $\lambda(\gamma - 1)$.

That is, given the optimal policy p_o, I explore the optimal policy p that the planner would design if he were given the opportunity to revise his policy. There are several remarks that should be made here. In problem (15), at the moment of designing the new policy, the planner takes the current active patent as given. That is, the new policy starts at the moment in which the next invention arrives. In concrete terms, the firs-race investments ($x_{i,t}$, x_n) are determined by the policy p_o and are not directly affected by the choice of p. Secondly, and related with the previous point, the planner can affect the first-race incentives through the value of a new patent v_o, which is now a function of the future policy p. Lastly, I assume that the firms and the planner ignore the possibility of future policy changes. The agents do not consider future policy changes when designing a patent system or when deciding the R&D investments. In formal words, what I study corresponds to a one-time unexpected change for both the planner and the firms.

Figures 5a and 5b show how the social welfare evolves in a new market for different values of the policy variables (T, β), under different values of λ.[12] An interior and unique optimal policy is typically found. In both scenarios, we can observe that the social cost of having the incorrect policy may be substantial. In the case of $\lambda = .5$, patent systems that are too short are the more detrimental to social welfare. On the other hand, when $\lambda = 1$, social welfare is harmed under policies that are either too weak or too strong.

Result 10 (Cost of incorrect policy) *The social cost of implementing the incorrect patent policy is increasing in the market's natural innovation rate.*

In Figure 5c I study how the optimal patent length reacts to changes in the natural innovation rate. In a new market, when λ is close to zero and innovations occur at a slow pace, the optimal patent length diverges to infinity. In this scenario, in order to speed up the innovation process, the planner needs to provide huge incentives for innovators to undertake the project, committing to very long patents. As the natural innovation rate increases, the optimal patent length decreases. This result is driven by two forces: i) since the generation of incentives through patent length is costly in terms of consumer surplus, when markets are naturally endowed with higher innovation rates the planner has a lower need to provide incentives for innovation, decreasing the patent length; ii) when the market's natural innovation rate is high, patent length is an ineffective tool to encourage innovation. This ineffectiveness is due to the fact that the gap between effective and statutory patent length increases under higher rates of innovation, inducing the planner to lose control over patent length as a policy tool. In particular, when λ goes to infinity, the optimal length converges to zero.

[12] Similar depictions can be obtained for a mature market.

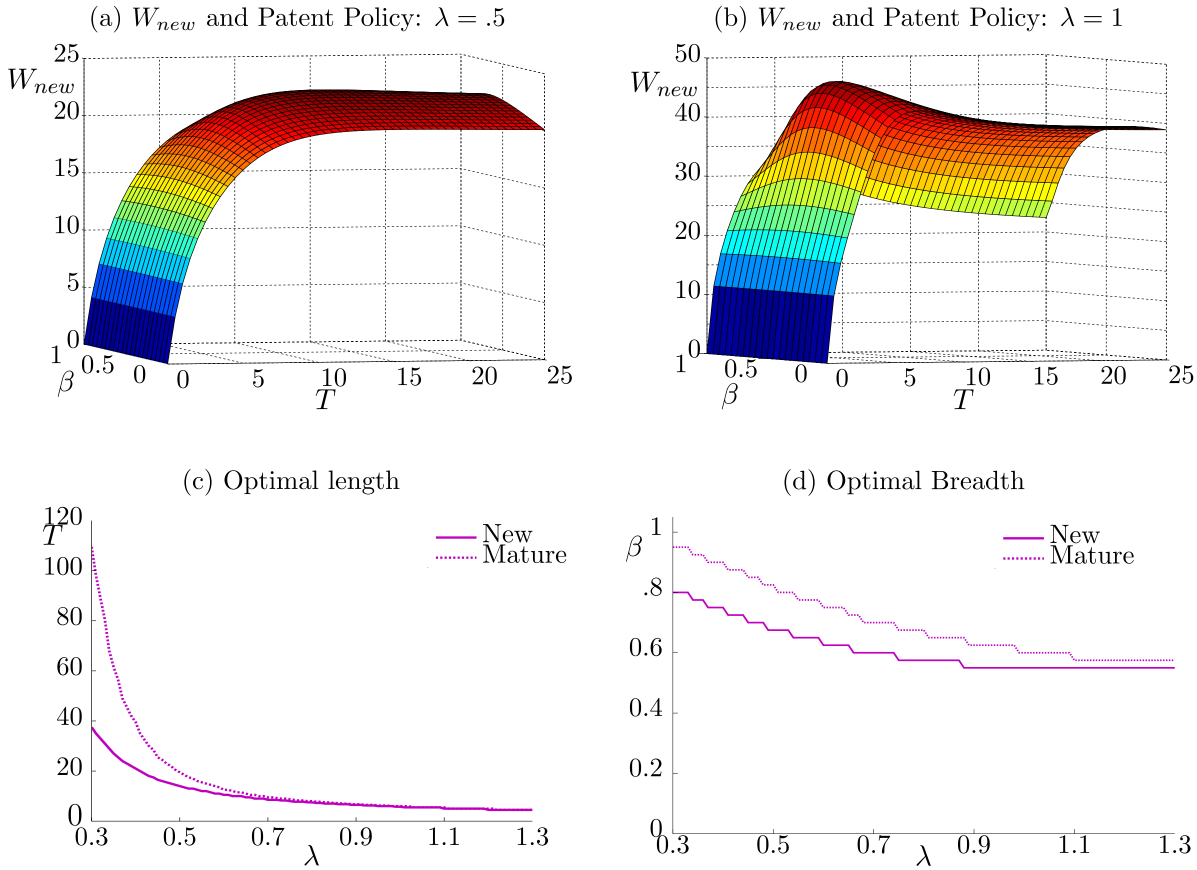

FIGURE 5: Welfare and optimal patent policy: $r = 0.05$ and $\gamma = 1.2$.

Result 11 (Optimal patent length–new market) *When no market exists, the optimal patent length is decreasing in the market's natural innovation rate regardless of the type of market. In the limit, optimal length converges to zero.*

Similarly, in a new market, optimal patent breadth decreases with the market innovation rate (see Figure 5d). Unlike the optimal length, when the arrival rate is high, the optimal breadth converges to a positive amount. When no market exists, there is no first-race cost associated to higher patent breadth, as infringement is not possible. On the contrary, higher breadth increases the value of a new patent, increasing the equilibrium investments. It is only after the market is created that breadth becomes costly for the planner, as higher breadth reduces the entrants' efforts. Thus, as the market innovation rate increases, the planner puts more weight to the subsequent races, decreasing patent breadth.

Breadth does not converge to zero in more innovative markets. The intuition for this result comes from the fact that in markets with large natural innovation rates, patent length loses its effectiveness, as the gap between effective and statutory patent length increases. As this effect becomes stronger, the only mechanism that a planner has to extend the patent's life and to effectively encourage innovation is to make the replacement of the incumbent's technology more difficult, i.e., to increase patent breadth.

Result 12 (Optimal patent breadth–new market) *When no market exists, the optimal breadth is decreasing in the market's natural innovation rate and converges to a positive amount for sufficiently high λ.*

In a mature market, the incentives that a planner faces when designing an optimal patent policy differ to those in a new market in two respects. First, if the planer were to implement p_0 in a mature market, the entrants competing in the first race would invest at a lower rate than the first-race entrants in a new market. Therefore, if the planer wants to generate similar investment rates from potential entrants, stronger policy is needed. On the other hand, the incumbent's incentives to innovate are a function of the prize that he obtains from innovating $v_o(p)-v(t,p,p_o)$. The value of the currently active patent $v(t,p,p_o)$ depends on t, on p_o, as it determines the length and the breadth of the patent, and on p, through the effect it has in v_o. If a policy p were such that $v_o(p) < v(0,p,p_o)$, the incumbent performs no effort until an instant t°, defined by $v(t^\circ,p,p_o) = v_o(p)$. From instant t° and on, the incumbent's investments increase with the approximation of the patent expiration date. Therefore, the only way to speed up the incumbent's innovation process is to provide a policy that grants greater protection than p_o. As can be seen in Figures 5c and 5d, these two effects complement each other, inducing the planner to adopt a stronger patent policy when given the chance to update it.

Result 13 (Optimal patent–mature) *In a mature market, the optimal patent policy implemented by the planner is stronger than the policy implemented in a new market. The difference in policies is decreasing in the market's innovation rate, and converges to zero when λ is sufficiently high.*

Finally, I study the effect of changes in γ (not depicted). When γ increases there are two complementary effects. First, there is an increase in the profits of the incumbent $(\gamma-1)/\gamma$. This leads to an increase in the value of a new patent and to more investments. Secondly, the gain from moving from a patented market to a competitive one, γ, increases. Both effects complement each other, decreasing both policy parameters.

Result 14 (Effect of a quality increase) *An increase in the quality improvement γ reduces both policy variables.*

6 Extensions

6.1 Royalties

In this section I study the alternative assumption that license fees are a fraction α of the flow profits π and are paid only while the infringement lasts, i.e. while the protection of the infringed patent is still active and the infringing firm keeps its incumbency status. The key difference, with respect to the original set up, is that the total expected license fees change throughout the patent life. In particular, the expected license fees decrease as the expiration date of the active patent approaches.

As before, we will have myopic entrants and an incumbent. Now, however, we may have an incumbent whose patent infringes the previous patent, in which case I denote the infringing firm by f, or an incumbent that does not infringe, i. Let $\tau \equiv T - t$ denote the time of protection that an infringed patent has left at the moment of the infringement at instant t. Then, the expected present value of the license fees at any instant $t < \tau$ is given by

$$l_t^\tau = \int_t^\tau \alpha\pi e^{-z_{t,s}^\tau} e^{-r(s-t)} ds.$$

That is, the infringed firm receives the discounted payoff $\alpha\pi$ at instant s, conditional on no innovation occurring between t and s, which happens with probability $\exp(-z_{t,s}^f)$. [13] In other words, when an innovation arrives, the payment of royalties cease as the profits derived from the infringing patent fall to zero. When τ time has passed by, the infringed patent loses its protection and no more license fees are collected, i.e. $l_\tau^\tau = 0$.

The value of a non-infringing patent at t is

$$v(t) = \max_{\{x_{i,s}\}_t^T} \int_t^T \left(\pi + \lambda x_{i,s} v_o + \lambda x_{n,s} b(l_0^{T-s}) - \frac{x_{i,s}^2}{2} \right) e^{-z_{t,s}} e^{-r(s-t)} ds. \tag{16}$$

This expression is similar to the value of a patent (1) in our previous set up. The key difference occurs when an entrant succeeds. If an entrant succeeds at instant s, his invention infringes on the incumbent's patent with probability b. In that case, the incumbent receives the expected license fees l_0^{T-s}.

On the other hand, since the the license fees are paid only while the infringed patent is protected, the value of an infringing patent depends on which moment of the infringed patent life the innovation occurred. Denote the value of an infringing patent at an instant

[13] Analogous to the previous notation: $z_{t,k}^\tau = \lambda \int_t^k (x_{f,s}^\tau + x_{n,s}) ds$, where $x_{f,s}^\tau$ is the effort performed at instant t, by an infringing incumbent that infringed the previous patent when it had τ time left.

$t < \tau$ by $m^{\tau}(t)$, then

$$m^{\tau}(t) = \int_{t}^{\tau} \left((1-\alpha)\,\pi + \lambda x_{f,s}^{\tau} v_o + \lambda x_{n,s} b(l_0^{T-t}) - \frac{(x_{f,s}^{\tau})^2}{2} \right) e^{-z_{t,s}} e^{-r(s-t)} ds$$
$$+ v\,(\tau)\, e^{-z_{t,\tau}} e^{-r(\tau-t)},$$

As before, the previous expression is similar in spirit to equation (1) but with two key differences. First, while license fees are paid, the incumbent receives a fraction $(1-\alpha)$ of the profits. Secondly, when instant τ is reached, his patent has the same value as a non-infringing patent at τ; hence, $m^{\tau}(\tau)$ has as a boundary condition $m^{\tau}(\tau) = v\,(\tau)$.

Using the maximum principle, it can be shown that the value of an infringing and non-infringing patent at instant t are

$$\begin{aligned} rv(t) &= \max \left\{ \pi + \lambda x_{i,t}\,(v_o - v\,(t)) + \lambda x_{n,t} \left(b l_0^{T-s} - v\,(t) \right) - \frac{x_{i,t}^2}{2} + v'\,(t) \right\} \\ rm^{\tau}(t) &= \max \left\{ (1-\alpha)\,\pi + \lambda x_{f,t}^{\tau}\,(v_o - m^{\tau}(t)) + \lambda x_{n,t} \left(b(l_0^{T-s}) - m^{\tau}(t) \right) - \frac{(x_{f,t}^{\tau})^2}{2} + (m^{\tau}(t))' \right\}. \end{aligned}$$

The first-order condition for the previous problems are

$$x_{i,t}^{*} = \lambda\,(v_o - v(t)) \text{ and } x_{f,t}^{*\tau} = \lambda\,(v_o - m^{\tau}(t))\,.$$

As before, both types of incumbents will have increasing efforts that are maximal at $t = T$. Since for all $\tau > 0$ we must have $f^{\tau}(t) < v(t)$, the effort of an infringing incumbent is greater than a non-infringing one for all $t < \tau$. At $t = \tau$, when the infringed patent protection expires and license fees are no longer paid, the efforts of both types of incumbents coincide.

Finally, the profits of an entrant at t are given by

$$\lambda x_{n,t}(bm^{T-t}(0) + (1-b)v_o) - x_{n,t}^2/2.$$

Since the entrants are myopic, they do not condition their efforts on the type of incumbent they face as the payoffs flow are independent of the opponent. Maximizing the entrant's payoff at each t we find that the optimal effort is

$$x_{n,t}^{*} = \lambda \left(v_o - b \left(v_o - m^{T-t}(0) \right) \right).$$

By construction, the value of being an infringing entrant at t, $m^{T-t}(0)$, is increasing in t as proximity to the infringed patent expiration date decreases the expected license fees. Therefore, now entrants' efforts are also increasing through time, reaching its maximum at $t = T$ when no infringement can occur (observe that $m^0(0) = v(0)$ by the boundary condition of the infringer's problem.) The investment dynamics are depicted in figure 6a, where the evolution of efforts for entrants, a non-infringing incumbent, and an infringing incumbent that innovated when the infringed patent had τ time left are shown.

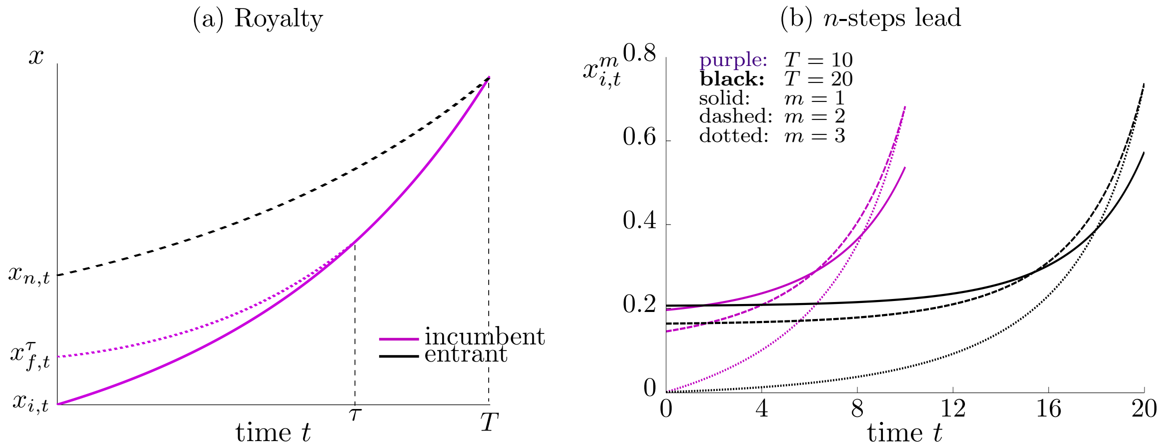

FIGURE 6: Extensions

6.2 n-Steps Lead

This section relaxes the assumption that the maximum lead an incumbent can have is a single technological step. There are three differences in the modeling assumptions of this section to those presented in section 2. First, for simplicity, I assume that two firms compete instead of assuming an incumbent and a sequence of short-run investors. As a consequence, this assumption implies that now we will have a value function describing the benefit of being an entrant facing an incumbent that posses an active patent at t. Secondly, the flow of profits depends on the size of the lead $m \in \mathbb{N}$ that the incumbent has over his competitors. Let π_m be the profit flow under a lead of size m. I assume $\pi_m > \pi_{m-1}$, implying that greater technological leads are valuable. Finally, and with sole purpose of having one state variable, I assume that when the incumbent succeeds he is able to extend the protection of his active patents to the expiration date of his latest discovery. This assumption magnifies the value of a new invention for an active incumbent, overestimating the effort he performs in equilibrium.

Let $v_m(t)$ denote the value of a patent that has a m-step lead and has been active for t years, and denote by $w_m(t)$ the value of being an entrant that faces an incumbent with a m-step technological lead with a patent that has been active for t years. Using the Principle of Optimality, the value of a m-step lead patent a t is

$$rv_m(t) = \max_{x_{i,t}}\{\pi_m - c\left(x^m_{i,t}\right) + \lambda x^m_{i,t}\left(v_{m+1}(0) - v_m(t)\right) \\ + \lambda x^m_{n,t}\left(w_1(0) + \beta v_1(0) - v_m(t)\right) + v'_m(t)\}.$$

There are three differences in the previous expression to that in equation (3). First, the incremental rent now takes into account that a new innovation increases the technology gap. Secondly, in case of being overcome by the entrant, the incumbent obtains the expected value of being a brand new entrant $w_1(0)$, and, with probability β, the payment from the possible infringement. Lastly, when the patent expiration date is reached, the continuation value of the project is equal to

$$V = \lambda^2 v_1(0) \frac{v_1(0) + 2w_1(0)}{4\lambda^2 v_1(0) + r}.$$

This is the value of being part a two-firm patent race, where both firms compete myopically and there is no risk of infringement.

The first order condition of the incumbent's Bellman equation is

$$x_{i,t}^m = \lambda(v^{m+1}(0) - v^m(t)).$$

Here, we can readily see that patent policy will affect both the value of a new patent with a $m+1$ technological gap, and the current value of the actual patent.

As before and for the sake of obtaining an analytical solution, I assume that the entrant chooses his effort myopically. Then the value of being an entrant facing an incumbent with an m-step technological lead with a patent that has been active for t years is

$$w_m(t) = \int_t^T \left(\lambda x_n^* v_1(0)(1-\beta) - c(x_n^*)\right) e^{-z_{t,s}^m} e^{-r(s-t)} ds + e^{-z_{t,T}^m} e^{-r(T-t)} V \tag{17}$$

where the strategy x_n^* is given by equation (2) and $z_{t,s}^m = \lambda \int_t^s \left(x_{i,x}^m + x_n\right) dx$. The previous equation can be explained as follows. The value of being an entrant at t is the expected discounted payoff induced by the effort rate x_n^* between the instant t and T, plus the continuation value V when the patent expires.

Following similar steps to those presented in Appendix A and using equation (17) to compute the values of $w_1(0)$ and V, an analytical solution for $v^m(t)$ can be obtained. Figure 6b depicts the incumbent's effort when the maximal technological gap is fixed to $m = 3$ under two different statutory lengths. There we can observe that with an increase in the technological gap, the incumbent's effort is lower at the beginning of the patent protection but becomes higher at the end of the protection. Also, an increase in patent length raises the effort at the very beginning of the patent protection but decreases it afterwards.

7 Relation with Previous Literature

The purpose of this section is to relate previous findings to the literature on patent policy and innovation. For clarity, the discussion is divided into a comparison of my results to those in the single-innovation literature, and to those in the sequential innovation literature. Finally, I discuss how this work relates to the literature of innovation and growth.

Single-innovation models and patent policy In the seminal work of Nordhaus (1969), inventions are produced by a deterministic production function under which innovations of higher quality have higher production costs. The main conclusion of Nordhaus's work is that longer patent protection induces inventions of higher quality but a greater social cost of monopoly. The interaction of these two effects calls for finite patent protection.

In contrast, my model belongs to a class of stochastic innovation. In these models inventions of a predetermined quality arrive stochastically and as a function of the firms' investment. Similar in spirit to Nordhaus, in the context of a single-innovation, Loury (1979) and Lee and Wilde (1980) find that longer patents increase the speed of innovation. Also, Denicolò (1999) makes the point that optimal patent length is finite due to the cost in consumer surplus. With respect to the single-innovation literature the key insight gained from analyzing the effects of patent length in a sequential world is that longer patents do not necessarily lead to higher innovation rates and therefore the cost associated with stronger patent policy may well be beyond the consumer surplus losses emphasized by the literature described above.

Another difference generated by the sequential framework is that incentives become non-stationary through time. In particular, investments increase as the patent expiration date approaches. In most of the single-innovation literature, patent races are stationary, meaning that a constant effort rate is performed by all the firms until an innovation occurs (an exception is Reinganum (1982) where the non-stationarity comes from the assumption that innovations may be only generated within a predetermined period of time). Even though the non-stationary property increases the difficulty of the analysis, it has the advantage of generating more realistic effort dynamics.

Earlier models of innovation, mainly inspired by Schumpeter (1942), studied the difference in incentives that incumbents and potential entrants may face (see Gilbert (2006) for a comprehensive theoretical and empirical review of the subject). Arrow (1962) showed that, when innovation cannibalizes part of an incumbent's profits, incumbents have less incentives to innovate than entrants. This effect is typically called the replacement effect. In contrast, Gilbert and Newbery (1982) argued that in an auction-like environment, where the firm that makes the highest investment/bid is the one that gets the patent, incumbents may be willing to outbid entrants due to the fact that the former may have more to lose than the latter to gain. One of my main contributions is to show that Arrow's effect is the key determinant of the incumbent's investment dynamics, and that patent policy not only affects the value of a new patent, it also affects the value of the replacement effect predicted by Arrow. Furthermore, is precisely the role of patent policy in determining the value of the replacement effect, which causes the decrease the economy's innovation rate for long patent protection.

Another branch of the single-innovation literature assumes that the costs and benefits of an innovation are known. In that respect, optimal patent policy consists of maximizing social welfare subject to an innovator-breaks-even constraint (Gilbert and Shapiro (1990); Klemperer (1990) and, in an environment with imitation, Gallini (1992)). This type of analysis is precluded in a sequential-stochastic world as both the cost of an innovation and the benefit of incumbency are random.

Policy in Sequential Innovation In spirit, the closest work to mine is Bessen and Maskin (2009). In a model of sequential innovation, where firms decide whether or not to undertake an innovative activity, they compare the situation in which either an infinitely long patent protection exists, or it does not. They find that patent protection may slow down the innovation process. The main contrast of their model to mine is that in my work I allow for patents of any length, not only the polar cases that deliver stationary investments. Also, in my framework, patents are necessary to promote innovation as the absence of patent protection drives the profits from an innovation to zero. In contrast, in their model, firms get positive profits regardless of the protection provided. Finally, while in their framework innovation does not cannibalize profits derived from previous innovations, my model is one of Creative Destruction where the cannibalization is total. In the substantive side, their result is driven by the existence of asymmetric information in the opponent's cost of R&D. In contrast, my result is driven by the role that patent policy plays in determining the incumbent's opportunity cost of innovation. Thus, both set of results complement each other by indicating different channels through which long patents may harm the innovation process.

Most sequential models focus on modeling different aspects of patent breadth (see Scotchmer (1991) for an insightful narrative paper that explains the general trade-offs explored in this literature). The early work of Scotchmer and Green (1990) and Scotchmer (1996) studies how patentability of second-generation products affects the development of first-generation products. Green and Scotchmer (1995) study the role of ex-ante and ex-post licensing agreements on R&D incentives.

A more recent literature initiated by O'Donoghue, Scotchmer and Thisse (1998) studies the effect of patents' novelty requirements on the market innovation rate in the context of an infinitely-lived market. These papers generally assume an infinitely-long patent protection and use patent breadth as a mechanism to deter inventions of lower quality. The lack of novelty requirements leads to fast imitation and to a reduction of the expected benefits of an innovation. Therefore, it is found that some breadth encourages innovation. Optimally, as shown by Hopenhayn *et al.* (2006), the minimal improvement required to obtain a patent is an increasing function of the quality of the previous innovation. I complement the study of patent systems by incorporating the effect of finite statutory lengths into the analysis.

My main contribution to the patent breadth literature is to highlight that its effectiveness depends on the patent statutory length and on the characteristics of the economic environment. In general, patent length is a more effective tool when the natural innovation rate of the market is low. Breadth becomes the predominant tool to encourage innovation when the gap between the the effective patent length and the statutory length renders patent length ineffective. This gap is a function of the market's natural innovation rate. In particular, when innovations are frequent by nature, patent length loses it effectiveness as innovations consistently arrive before the patents' statutory length. In that circumstance, breadth becomes the predominant tool as helps the planner to extend the effective length of a patent by making harder to overcome the incumbent's innovation.

More recently, there has been effort in understanding how other elements of policy affect innovation. For instance, Segal and Whinston (2007) study how antitrust affects the rate of innovation in a given market. My work complements this literature by deepening the understanding of how the different policy tools available can affect innovation.

Innovation, Market Structure and Growth In order to study how patent policy affects social welfare I assume a quality ladder growth model. These models are generally used to study how different elements of the economy affect economic growth in a context where growth is mainly driven by the arrival of new innovations (see for example Grossman and Helpman (1991), and Aghion and Howitt (1992).) Even though the focus of my work is not to study how patent policy affects economic growth, my research provides a framework to investigate the role of patent policy in growth. In particular, my restriction of looking at one particular market instead of an economy as a whole is now both feasible and worthy of exploration.

8 Conclusion

This work developed a tractable model of sequential innovation, and studied how patent policy affects the R&D investment dynamics in an economy described by a quality ladder and by the Shumpeterian property of creative destruction. Due to profit cannibalization, I found that incumbents delay their investment towards the end of their patent protection as the benefit of an innovation increases with the approximation of the patent expiration date. This result is consistent and provides an explanation to the empirical evidence that incumbents: invest less in R&D, patent less, and adopt new technologies later.

I found that patent policy plays an important role in determining the rents of an innovation. In particular, patent policy not only determines the value of obtaining a new patent, it also determines the value of replacing the active patent held by the incumbent. For instance, an increase in patent length increases the value of an active patent more than the value of a new patent, decreasing the incumbent's prize from an innovation. As a consequence, at each point in time, the incumbent's effort rate decreases when offered longer protection, inducing the economy's innovation rate to react non-monotonically to changes in patent length. In particular, a patent extension increases the economy's innovation rate when the initial statutory length of a patent is short, and the extension decreases the economy's innovation rate when the initial statutory length is long. Therefore, the common result that stronger patent protection encourages growth is shown not to apply to this dynamic setting.

From a policy perspective, I show that the optimal incentives needed to induce the creation of new products and those needed to induce improvements of existent products differ. In the former case, higher levels of breadth are desirable, whereas in the latter the opposite is true. I also show that optimal patent policy varies substantially with market characteristics such as the market's natural innovation rate. This poses the questions of how policy makers can measure this variable and how the actual policy can embed this dependence in the current framework.

Important questions about how patent policy affects innovation in a sequential context remain open. For instance, firms do not always disclose their inventions right after developing them. For incumbent's, this may be done to avoid cannibalization of their profit flow. Entrants, on the other hand, may do it to avoid potential license fees from soon-to-be-expired patents. These potential delays of disclosure come at a cost; if the competition were to release the new technology first, the firm that waited may lose its potential benefit of being the next incumbent. Patent systems play an important role in the disclosure of innovations. If two firms where to claim similar technology to the patent office, in a first-to-invent system, the office will rule in favor of the innovator that invented a patent first but chose to not disclose his invention. In contrast, in a first-innovator-to-file system, the patent office will tend to rule in favor of the first firm that arrived at the patent office. I believe that my model could provide a basis to begin answering this or other questions about sequential innovation.

Appendices

A Solution to ODE

In this appendix I solve the differential equation that describes how the value of having a patent evolves with the proximity of its expiration date. For notation simplicity, I restate the differential equation (5) as

$$\frac{dv}{dt} + a_0 v^2 - a_1 v + a_2 = 0$$

where

$$a_0 = \frac{\lambda^2}{2},\ a_1 = r + \lambda^2 (2-\beta)\, v_o,\ \text{and } a_2 = \pi + \lambda^2 \left(\frac{1}{2} + (1-\beta)\,\beta\right) v_o^2.$$

This ODE is separable and of the form $dv/h(v) = dt$ where $h(v) = -\left(a_0 v^2 - a_1 v + a_2\right)$. Separable ODEs have unique non-singular solution that goes through its boundary condition, in this case $v(T) = 0$. [14] To find the non-singular solution I integrate both sides to get

$$\ln\left(\frac{a_1 - 2v_t\theta_0 + \sqrt{a_1^2 - 4a_0a_2}}{a_1 - 2v_t a_0 - \sqrt{a_1^2 - 4a_0a_2}}\right)\sqrt{\frac{1}{a_1^2 - 4a_0a_2}} = \hat{C} + t$$

where $\hat{C}$ is the constant of integration. Define $\phi = \sqrt{a_1^2 - 4a_0a_2}$ and solving for v_t we find

$$v_t = \frac{1}{2a_0}\left(a_1 + \phi\frac{\left(1 + e^{-\phi\left(\hat{C}+t\right)}\right)}{\left(1 - e^{-\phi\left(\hat{C}+t\right)}\right)}\right). \tag{18}$$

which is the general solution to the ODE. To find the particular solution we just make use of the boundary condition $v(T) = 0$

$$\hat{C} = -\frac{1}{\phi}\ln\left(\frac{a_1+\phi}{a_1-\phi}\right) - T. \tag{19}$$

Replacing back (19) in to (18) and rearranging terms we obtain

$$v(t, v_o) = \frac{1}{\lambda^2}\left(\frac{\left(\theta^2 - \phi^2\right)\left(e^{\phi(T-t)} - 1\right)}{\theta\left(e^{\phi(T-t)} - 1\right) + \phi\left(e^{\phi(T-t)} + 1\right)}\right)$$

[14] Singular solutions to (5) are found by setting $dv/dt = 0$ and solving the quadratic equation. These solutions are disregarded as they do not generically satisfy the boundary condition $v(T) = 0$ and have no economic meaning.

which corresponds to equation (6). Finally, I make sure that the function $v\left(t, v_o\right)$ is well defined for all positive values of v_o. This clearly is true when v_o is such that $\phi > 0$. I have to check the cases under which ϕ is either imaginary and zero. For the former, let $\phi = qi$ where q is the positive real coefficient of i. Rewrite $v\left(t, v_o\right)$ as

$$v\left(t, v_o\right) = \frac{1}{\lambda^2}\left(\frac{\left(\theta^2 - \phi^2\right)}{\theta + \phi\frac{e^{\phi(T-t)}+1}{e^{\phi(T-t)}-1}}\right)$$

and observe that Euler's identity implies[15]

$$\phi\frac{e^{\phi(T-t)}+1}{e^{\phi(T-t)}-1} = \frac{q\sin\left(q\left(T-t\right)\right)}{1-\cos\left(q\left(T-t\right)\right)} \tag{20}$$

so the value of a patent $v\left(t\right)$ is real.

Finally, let $\hat{v}$ be the value of v_o such that $\phi\left(\hat{v}\right) = 0$. When $\phi = 0$ the value of the patent at every t becomes $v\left(t\right) = 0/0$. Thus, I take as the value of $v\left(t\right)$ to be $\lim_{v_o \to \hat{v}} v\left(t\right)$ which can be computed through L'Hôspital's rule and is equal to

$$\frac{1}{\lambda^2}\frac{\theta^2\left(T-t\right)}{2+\theta\left(T-1\right)}$$

proving that $v\left(t, v_o\right)$ is well defined.

B Omitted Proofs

B.1 Proof of Proposition 1

I start by proving existence. From appendix A we know that there is a unique solution of the ODE, so I can restrict attention to show that exists a fixed-point $v(0, v_0) = v_o$ for positive values of v_o.[16] To do so, start by reformulating the problem defining $f(x) = v(0,x) - x$ and then show there exists a solution to $f(x) = 0$ where

$$f(x) = \frac{1}{\lambda^2}\left(\frac{\left(\mu-\phi\right)\left(\theta+\phi\right)e^{\phi T} - \left(\theta-\phi\right)\left(\mu+\phi\right)}{\theta\left(e^{\phi T}-1\right)+\phi\left(e^{\phi T}+1\right)}\right) \tag{21}$$

and

$$\phi = \sqrt{\theta^2 - \lambda^2\hat{\theta}}, \quad \theta = r + \lambda^2\left(2-\beta\right)x, \quad \mu = r + \lambda^2(1-\beta)x \text{ and}$$
$$\hat{\theta} = 2\pi + \lambda^2\left(1 + 2\left(1-\beta\right)\beta\right)x^2.$$

[15] Euler's identity: $e^{i\psi} = \cos(\psi) + i\sin(\psi)$

[16] There may be other fixed points such that $v_o < 0$; however, those do not have an economic meaning and, consequently, are ignored

where all the previous coefficients are function of the fixed-point x.

Existence is shown using the intermediate value theorem. Start by observing that ϕ goes to ∞ when x goes to infinite (see appendix C). Then, is easy to check that $\lim_{x\to\infty}(\mu-\phi)=-\infty$ and therefore $\lim_{x\to\infty} f(x)=-\infty$. I now show that exist x such that $f(x)>0$. The initial guess is $x=0$; however, ϕ is imaginary at that point if $r^2<2\lambda^2\pi$. Hence, I divide the proof in two cases.

case 1 ($r^2\geq 2\lambda^2\pi$): Here ϕ is a positive real for $x\geq 0$ and

$$f(0)=\left(\frac{2\pi\left(e^{\phi T}-1\right)}{\theta\left(e^{\phi T}-1\right)+\phi\left(e^{\phi T}+1\right)}\right)>0.$$

Proving existence for this set of parameters.

case 2 ($r^2<2\lambda^2\pi$): Here ϕ is imaginary for all $x\in[0,x^\circ)$ where $x^\circ>0$ is the one that satisfies $\phi(x^\circ)=0$. At that point $f(x^\circ)=\frac{0}{0}$ and using the limit value of $v(0,x^\circ)$ provided in Appendix A we obtain

$$f(x^\circ)=\frac{-2\lambda^2x^\circ+T\theta\mu}{2+T\theta}$$

where θ and μ are positive at every positive x, in particular at $x^\circ>0$. It is clear that $\lim_{x\to x^\circ} f(x^\circ)$ is positive for $T>T^*$ where

$$T^*=2\lambda^2x^\circ/(\theta\mu) \tag{22}$$

completing the proof for those values of T. To give a proof for the case $T\leq T^*$ write $f(x)$ using equation (20) in Appendix A

$$f(x)=\frac{1}{\lambda^2}\frac{(\theta\mu+q^2)(1-\cos(Tq))-\lambda^2xq\sin(Tq)}{\theta(1-\cos(Tq))+q\sin(Tq)}. \tag{23}$$

where q is the positive coefficient of $\phi=qi$. If $\sin(Tq)>0$ at $x=0$, then

$$f(0)=2\pi\frac{(1-\cos Tq)}{r(1-\cos(Tq))+q\sin(Tq)}\geq 0.$$

If $\sin(Tq)<0$ at $x=0$, then choose the value of x^* such that $\sin(Tq)=0$ and $\cos(Tq)=-1$. This number exists as $q(x)T\in[0,q(0)T]$ for values of $x\in[0,x_o]$. Then

$$f(x^*)=\frac{1}{\theta\lambda^2}\left(q^2+\theta\mu\right)>0$$

proving existence.

Finally, effort at t is found by replacing $v(t)$ in equation (7) and is increasing as

$$\frac{\partial v(t)}{\partial t}=-\frac{2\phi^2(\theta^2-\phi^2)e^{\phi(T-t)}}{\lambda^2\left((\theta+\phi)e^{\phi(T-t)}-(\theta-\phi)\right)^2}<0.$$

B.2 Proof of Proposition 2

I start by showing that the direct effect of an increase in the statutory length T on $v(t)$ is positive for all t (i.e. ignoring that a change in T also changes the fixed-point v_o)

$$\frac{\partial v(t)}{\partial T}=\frac{2\phi^2\left(\theta^2-\phi^2\right)e^{\phi(T-t)}}{\lambda^2\left(\left(\theta+\phi\right)e^{\phi(T-t)}-\left(\theta-\phi\right)\right)^2}>0. \tag{24}$$

In particular, at $t=0$ the direct effect is the net effect and $v(0)$ is increasing in T. For the other comparative statics observe

$$v(0)=\int_0^T\left(\pi+\lambda x_{i,t}^*v_o+\lambda x_n^*b\alpha v_o-c\left(x_{i,t}^*\right)\right)e^{-z_{0,t}}e^{-rt}dt.$$

The direct effect of an increase in π is equal to

$$\frac{\partial v(0)}{\partial\pi}=\int_0^T e^{-z_{0,t}}e^{-rt}dt>0 \tag{25}$$

which is clearly positive implying the result. The indirect effect of an increase in competition, caused by the increase in v_o, cannot overcome the increase in valuation, this would be a contradiction to the fact that the increase in competition was caused by the increase in v_o in the first place. Similarly, an increase in the interest rate r

$$\frac{\partial v(0)}{\partial r}=-\int_0^T t\left(\pi+\lambda x_{i,t}^*v_o+\lambda x_n^*b\alpha v_o-c\left(x_{i,t}^*\right)\right)e^{-z_{0,t}}e^{-rt}dt<0 \tag{26}$$

which is positive, implying that an increase in the interest rate decrease the value of a new patent.

B.3 Proof of Proposition 5

Proposition 5.1 is direct consequence of Proposition 2.1, an increase in T increases v_o and, consequently, x_n. For 5.2 start by observing

$$\frac{\partial x_{i,t}}{\partial T}=\lambda\left(\frac{\partial v(0)}{\partial T}-\left(\frac{\partial v(t)}{\partial T}+\frac{\partial v(t)}{\partial v_o}\frac{\partial v_o}{\partial T}\right)\right)$$

That is, an increase in T has a direct effect at both $t=0$ and at $t>0$. Also, for all $t>0$ there is an indirect effect due to the increase in the fixed-point v_o.

Proposition 5.2 corresponds to the case under which we fixed $\partial v(t)/\partial T$ to zero as there is extension of the patent life for the current patent, that is

$$\frac{\partial x_{i,t}}{\partial T}=\lambda\frac{\partial v(0)}{\partial T}\left(1-\frac{\partial v(t)}{\partial v_o}\right)>0$$

where I have used $v(0) = v_o$. The inequality follows from the fact that the increase in $v(t)$ (a fraction of $v(0)$) due to an increase in v_o, cannot be more than the increase in $v(0)$ it self. Therefore, $\partial v(t)/\partial v_o < \partial v(0)/\partial v_o = 1$.

Finally, for Proposition 5.3 the whole effect has to be considered. Rewrite the derivative as

$$\frac{\partial x_{i,t}}{\partial T} = \lambda \left(\frac{\partial v(0)}{\partial T} - \frac{\partial v(t)}{\partial T} \right) - \frac{\partial v(t)}{\partial v_o} \frac{\partial v_o}{\partial T}$$

Since $(\partial v(t)/\partial v_o)(\partial v_o/\partial T)$ is by construction positive, is enough to show that the term in parenthesis is negative. This follows from the inequality (24) and

$$\frac{\partial^2 v(t)}{\partial T \partial t} = \frac{2}{\lambda^2} \phi^3 \frac{\left(\theta^2 - \phi^2\right) \left(\left(\theta + \phi\right) e^{\phi(T-t)} + \theta - \phi \right)}{\left(\left(\theta + \phi\right) e^{\phi(T-t)} - \left(\theta - \phi\right) \right)^3} e^{\phi(T-t)} > 0.$$

B.4 Proof of Proposition 7

I start by showing that the limiting value of a patent is given by equation (9). Taking the limit of (6) when T goes to infinity delivers

$$\lim_{T \to \infty} v\left(t\right) = \frac{1}{\lambda^2} \left(\theta - \phi\right).$$

Since this is true for all t it has to be true for $v(0) = v_o$. Substituting at $t = 0$, we obtain a quadratic equation that has as a unique positive solution

$$\frac{1}{2\lambda^2 \left(1 - \beta\right)^2} \left(-r + \sqrt{r^2 + 4\pi\lambda^2 \left(1 - \beta\right)^2} \right)$$

which is the expression in equation (9), proving the second statement. To check that effort converges to zero take the limit of (7) to obtain

$$\lim_{T \to \infty} x_{i,t} = \frac{1}{\lambda} \left(\mu - \phi\right).$$

Substituting back the value v_∞ delivers the result.

C List of Derivatives and Integrals

For completeness, in this section I present some computations and a list of the derivatives and integrals used in the text.

The value of ϕ goes to infinity as v_o goes to infinity since it can be written as

$$\phi = \sqrt{3\lambda^4(1-\beta)x^2 + 2\lambda^2 r(2-\beta)x + r^2 - 2\lambda^2\pi},$$

since the coefficients of x are always positive, ϕ diverges when $x \to \infty$.

Derivatives

$$\begin{aligned}
\frac{\partial\phi}{\partial x} &= 2\lambda^2\frac{\left(3x\lambda^2\left(1-\beta\right)^2 + r\left(2-\beta\right)\right)}{\phi} > 0 \\
\frac{\partial\phi}{\partial\pi} &= -\frac{\lambda^2}{\phi} < 0 \\
\frac{\partial\phi}{\partial r} &= \frac{r + x\lambda^2\left(2-\beta\right)}{\phi} > 0 \\
\frac{\partial\phi}{\partial\beta} &= -x\lambda^2\frac{r + 3x\lambda^2\left(1-\beta\right)}{\phi} < 0
\end{aligned}$$

Integrals For ease in notation define $f = \theta - \phi$, $d = \theta + \phi$. Then

1. $z_{0,t}$

$$-\left(r+\phi\right)t - 2\ln\left(\frac{de^{\phi(T-t)} - f}{de^{\phi T} - f}\right)$$

2. $\int_0^T e^{-z_{0,s}}e^{-rs}ds$

$$\frac{(d^2e^{\phi T} + f^2)(e^{\phi t} - 1) - 2dfT\phi e^{\phi T}}{\phi\left(de^{\phi T} - f\right)^2}$$

3. $\int_0^T x_{i,s}e^{-z_{0,s}}e^{-rs}ds$

$$\frac{(d^2(\phi-\mu)e^{\phi T} - f^2(\phi+\mu))(e^{\phi T} - 1) + 2dfT\mu\phi e^{\phi T}}{\phi\left(de^{\phi T} - f\right)^2}$$

4. $\int_0^T (x_{i,s})^2e^{-z_{0,s}}e^{-rs}ds$

$$\frac{(d^2(\phi-\mu)^2e^{\phi T} + f^2(\phi+\mu)^2)(e^{\phi T} - 1) + 2dfT\phi(\phi^2-\mu^2)e^{\phi T}}{\phi\left(de^{\phi T} - f\right)^2\lambda^2}$$

References

Acemoglu, D., Bimpikis, K. and Ozdaglar, A. (2011). Experimentation, patents, and innovation. *American Economic Journal: Microeconomics*, **3** (1), 37–77.

Acs, Z. J. and Audretsch, D. B. (1988). Innovation in large and small firms: An empirical analysis. *American Economic Review*, **78** (4), 678–90.

Aghion, P., Bloom, N., Blundell, R., Griffith, R. and Howitt, P. (2005). Competition and innovation: An inverted-u relationship. *The Quarterly Journal of Economics*, **120** (2), pp. 701–728.

—, Harris, C., Howitt, P. and Vickers, J. (2001). Competition, imitation and growth with step-by-step innovation. *The Review of Economic Studies*, **68** (3), pp. 467–492.

— and Howitt, P. (1992). A model of growth through creative destruction. *Econometrica*, **60** (2), pp. 323–351.

Arrow, K. (1962). Economic welfare and the allocation of resources for invention. In *The Rate and Direction of Inventive Activity: Economic and Social Factors*, NBER Chapters, National Bureau of Economic Research, Inc, pp. 609–626.

Bessen, J. and Hunt, R. M. (2007). An empirical look at software patents. *Journal of Economics & Management Strategy*, **16** (1), 157–189.

— and Maskin, E. (2009). Sequential innovation, patents, and imitation. *The RAND Journal of Economics*, **40** (4), pp. 611–635.

Bonatti, A. and Hörner, J. (2011). Collaborating. *American Economic Review*, **101** (2), 632–63.

Bound, J., Cummins, C., Griliches, Z., Hall, B. and Jaffe, A. (1984). Who does r&d and who patents? In *R & D, Patents, and Productivity*, NBER Chapters, National Bureau of Economic Research, Inc, pp. 21–54.

Chang, H. F. (1995). Patent scope, antitrust policy, and cumulative innovation. *The RAND Journal of Economics*, **26** (1), pp. 34–57.

Choi, J. P. (1991). Dynamic r&d competition under "hazard rate" uncertainty. *The RAND Journal of Economics*, **22** (4), pp. 596–610.

Cohen, W. M. (2010). Chapter 4 - fifty years of empirical studies of innovative activity and performance. In B. H. Hall and N. Rosenberg (eds.), *Handbook of The Economics of Innovation, Vol. 1*, *Handbook of the Economics of Innovation*, vol. 1, North-Holland, pp. 129 – 213.

Czarnitzki, D. and Kraft, K. (2004). An empirical test of the asymmetric models on innovative activity: who invests more into r&d, the incumbent or the challenger? *Journal of Economic Behavior & Organization*, **54** (2), 153–173.

D'Aspremont, C. and Jacquemin, A. (1988). Cooperative and noncooperative r & d in duopoly with spillovers. *The American Economic Review*, **78** (5), pp. 1133–1137.

Denicolò, V. (1996). Patent races and optimal patent breadth and length. *The Journal of Industrial Economics*, **44** (3), pp. 249–265.

Denicolò, V. (1999). The optimal life of a patent when the timing of innovation is stochastic. *International Journal of Industrial Organization*, **17** (6), 827–846.

Denicolò, V. (2000). Two-stage patent races and patent policy. *The RAND Journal of Economics*, **31** (3), pp. 488–501.

Denicolò, V. and Zanchettin, P. (2002). How should forward patent protection be provided? *International Journal of Industrial Organization*, **20** (6), 801–827.

Doraszelski, U. (2003). An r&d race with knowledge accumulation. *The RAND Journal of Economics*, **34** (1), pp. 20–42.

Farrell, J. and Shapiro, C. (2008). How strong are weak patents? *The American Economic Review*, **98** (4), pp. 1347–1369.

Gallini, N. T. (1992). Patent policy and costly imitation. *The RAND Journal of Economics*, **23** (1), pp. 52–63.

Gilbert, R. (2006). Looking for mr. schumpeter: Where are we in the competition-innovation debate? In *Innovation Policy and the Economy, Volume 6*, NBER Chapters, National Bureau of Economic Research, Inc, pp. 159–215.

— and Newbery, D. M. G. (1982). Preemptive patenting and the persistence of monopoly. *The American Economic Review*, **72** (3), pp. 514–526.

— and Shapiro, C. (1990). Optimal patent length and breadth. *The RAND Journal of Economics*, **21** (1), pp. 106–112.

Green, J. R. and Scotchmer, S. (1995). On the division of profit in sequential innovation. *The RAND Journal of Economics*, **26** (1), pp. 20–33.

Grossman, G. M. and Helpman, E. (1991). Quality ladders in the theory of growth. *Review of Economic Studies*, **58** (1), 43–61.

Harris, C. and Vickers, J. (1987). Racing with uncertainty. *The Review of Economic Studies*, **54** (1), pp. 1–21.

Hopenhayn, H., Llobet, G. and Mitchell, M. (2006). Rewarding sequential innovators: Prizes, patents, and buyouts. *Journal of Political Economy*, **114** (6), 1041–1068.

Hunt, R. M. (2004). Patentability, industry structure, and innovation. *The Journal of Industrial Economics*, **52** (3), pp. 401–425.

Igami, M. (2011). Estimating the innovators dilemma: Structural analysis of creative destruction in the hard disk drive industry, manuscript, Yale University.

Katz, M. L. and Shapiro, C. (1985). On the licensing of innovations. *The RAND Journal of Economics*, **16** (4), pp. 504–520.

— and — (1987). R and d rivalry with licensing or imitation. *The American Economic Review*, **77** (3), pp. 402–420.

Keller, G., Rady, S. and Cripps, M. (2005). Strategic experimentation with exponential bandits. *Econometrica*, **73** (1), 39–68.

Klemperer, P. (1990). How broad should the scope of patent protection be? *The RAND Journal of Economics*, **21** (1), pp. 113–130.

Lee, T. and Wilde, L. L. (1980). Market structure and innovation: A reformulation. *The Quarterly Journal of Economics*, **94** (2), pp. 429–436.

Loury, G. C. (1979). Market structure and innovation. *The Quarterly Journal of Economics*, **93** (3), pp. 395–410.

Malueg, D. A. and Tsutsui, S. O. (1997). Dynamic r&d competition with learning. *The RAND Journal of Economics*, **28** (4), pp. 751–772.

Mukherjee, A. and Pennings, E. (2004). Imitation, patent protection, and welfare. *Oxford Economic Papers*, **56** (4), pp. 715–733.

Nordhaus, W. (1969). *Invention growth, and welfare: a theoretical treatment of technological change*. M.I.T. monographs in economics, M.I.T. Press.

O'Donoghue, T. (1998). A patentability requirement for sequential innovation. *The RAND Journal of Economics*, **29** (4), pp. 654–679.

—, Scotchmer, S. and Thisse, J.-F. (1998). Patent breadth, patent life, and the pace of technological progress. *Journal of Economics & Management Strategy*, **7** (1), 1–32.

Reinganum, J. F. (1982). A dynamic game of r and d: Patent protection and competitive behavior. *Econometrica*, **50** (3), pp. 671–688.

— (1983). Uncertain innovation and the persistence of monopoly. *The American Economic Review*, **73** (4), pp. 741–748.

— (1985). Innovation and industry evolution. *The Quarterly Journal of Economics*, **100** (1), pp. 81–99.

Sakakibara, M. and Branstetter, L. (2001). Do stronger patents induce more innovation? evidence from the 1988 japanese patent law reforms. *The RAND Journal of Economics*, **32** (1), pp. 77–100.

Salant, S. W. (1984). Preemptive patenting and the persistence of monopoly: Comment. *The American Economic Review*, **74** (1), pp. 247–250.

Schumpeter, J. A. (1942). *Capitalism, socialism and democracy / by Joseph A. Schumpeter*. George Allen & Unwin, London :, 4th edn.

Scotchmer, S. (1991). Standing on the shoulders of giants: Cumulative research and the patent law. *The Journal of Economic Perspectives*, **5** (1), pp. 29–41.

— (1996). Protecting early innovators: Should second-generation products be patentable? *RAND Journal of Economics*, **27** (2), 322–331.

— and Green, J. (1990). Novelty and disclosure in patent law. *The RAND Journal of Economics*, **21** (1), pp. 131–146.

Segal, I. and Whinston, M. D. (2007). Antitrust in innovative industries. *American Economic Review*, **97** (5), 1703–1730.

Segerstrom, P. S. (1991). Innovation, imitation, and economic growth. *Journal of Political Economy*, **99** (4), pp. 807–827.