Dynamic Persuasion*

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Abstract

I develop a model of dynamic persuasion. A sender has a fixed number of pieces of hard evidence that contains information about the quality of his proposal, each of which is either favorable or unfavorable. The sender may try to persuade a decision maker (DM) that she has enough favorable evidence by sequentially revealing at most one piece at a time. Presenting evidence is costly for the sender and delaying decisions is costly for the DM. I study the equilibria of the resulting dynamic communication game. The sender effectively chooses when to give up persuasion and the DM decides when to make decision. Resolving the strategic tension requires probabilistic behavior from both parties. Typically, the DM will accept the sender's proposal even when she knows that the sender's evidence may be overall unfavorable. However, in a Pareto efficient equilibrium, the other type of error does not occur unless delays costs are very large. Furthermore, the sender's net gain from engaging in persuasion can be negative on the equilibrium path, even when persuasion is successful. I perform comparative statics in the costs of persuasion. I also characterize the DM's optimal stochastic commitment rule and the optimal non-stochastic commitment rule; compared to the communication game, the former yields a Pareto improvement whereas the latter can leave even the DM either better or worse off.

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1 Introduction

Persuasion is the act of influencing someone to undertake a particular action, or, more generally, to form a certain belief. Successful persuasion takes time and is costly for both parties: the speaker exerts effort to present convincing arguments or information and, in turn, the listener reflects upon or inspects these carefully. A typical process of persuasion may involve a back-and-forth interaction where the speaker gradually presents a series of arguments up until when the listener is either sufficiently convinced by the speaker or has decided that the speaker's case lacks merit.

This paper is an attempt to understand some essential features of the dynamics of persuasion. To fix ideas, consider an example where an entrepreneur is trying to convince a venture capitalist (VC) to invest in his startup. The VC only wants to invest if the startup is sufficiently likely to succeed. The onus is on the entrepreneur to explain and validate a number of different aspects of the project that justify investment. Of course, the VC will scrutinize each argument, possibly hiring third parties to do so. In a stylized way, the process may unfold as follows: the entrepreneur presents a set of facts about the project that the VC scrutinizes, and then the VC decides to either invest, walk away, or request further explanation; and the process repeats, with the entrepreneur deciding whether to comply or give up on persuading the VC.

Observe that this process has a dynamic element of "matching pennies." If the VC knows that the entrepreneur only brings profitable plans, she just rubber-stamps his proposals, rather than paying the various costs incurred to scrutinize the plans. On the other hand, once the entrepreneur thinks that the VC will not carefully scrutinize his proposal, he may bring even sketchy plans. This, in turn, generates the VC's incentive to carefully scrutinize. This simple story, which has the flavor of the matching pennies game, shows that each player has an incentive to outfox his or her opponent.

Having this salient nature of persuasion in mind, this paper describes the dynamic process of persuasion in a formal game theoretic model. A sender (persuader, speaker) may try to persuade a decision maker (receiver, listener) that she has enough favorable evidence for his proposal by sequentially communicating evidence by paying the communication cost. He can also remain to be silent, which incurs no cost for him. At each period, the decision maker chooses whether to require another piece of evidence that delays her decision making, or not. Hence she chooses to require evidence as long as she can expect that there is an informational gain from doing so. We show that the equilibrium involves probabilistic decision making from both parties. The decision maker may make a decision before she gets enough information from the sender, so she may make the wrong decision.

Our model succeeds in providing some essential features of the dynamics of persuasion. Each time the sender communicates a piece of evidence, the decision maker updates her belief about the sender's proposal, and she accepts the proposal with a strictly positive probability. As the game proceeds, the decision maker accumulates more and more information and the probability that she makes the wrong decision decreases. A good sender, who has enough good evidence, continues to persuade by showing evidence until his proposal is accepted. However, he may pay too much communication costs so he may lose ex-post even though he was successful at persuasion. A bad sender tries to persuade the decision maker with some probability. When the decision maker judges that the cost of requiring a further piece of evidence exceeds the additional informational benefit, she accepts the proposal for sure and the process of persuasion terminates.

The fact that the equilibrium involves probabilistic decision-making stems from the game's similarity to the matching pennies game. If the decision maker does not accept the sender's proposal until a certain amount of evidence is shown, then the sender never tries to persuade unless he has enough good evidence. However this implies that the first piece of good evidence already screens out the bad sender and the decision maker loses the incentive to check the rest of the evidence. If the decision maker cannot make a "commitment to listen", they should use mixed strategy in order to get around this strategic tension, as in the matching pennies game.

General characterization of the equilibrium demonstrates the following results. There is a lower bound of the probability of immediate acceptance every time the sender communicates evidence (either good or bad). Actually this lower bound is the acceptance probability that makes the sender's communication cost equal his immediate expected gain. Also, silence never meets immediate acceptance, which tells us that only costly message has a persuasive power.[1] Finally, and rather obviously, the decision maker accepts the proposal for sure only after the sender shows a good piece of evidence.

Although there are a plethora of equilibria, we characterize the set of Pareto equilibria that are not Pareto dominated by other equilibria, and furthermore the best equilibrium for the decision maker, which is unique. Generally, it is possible to have an equilibrium that involves intuitively inessential stages such as the sender remains silent, which incurs no cost for him, and the decision maker just waits for the sender starts talking. We show that any Pareto efficient equilibrium excludes such redundant stages. Specifically, we show that in a Pareto efficient equilibrium, the sender never communicates bad evidence, sender's silence meets immediate rejection, and the decision maker's acceptance probability immediately after seeing a piece of good evidence is either maximized or minimized among all possible ways of constructing an equilibrium. It is also shown that in a Pareto efficient equilibrium, once the process of persuasion starts, the decision maker does not make the error of rejecting a good proposal. We further show that in the best equilibrium the decision maker requires the largest number of good evidence in order to accept for sure: intuitively, increasing the amount of good evidence necessary for persuasion discourages bad senders from trying to persuade.

The uniqueness of the best equilibrium for all parameter values enables us to pin down a reasonable benchmark on which we conduct comparative static analysis. We particularly examine the effects of two players' costs of communication on their expected payoffs and expected duration of persuasion. We show that a decrease in the costs of communication for the decision maker (delay costs) benefits her through two effects. The first one is the direct effect. The second one, which is indirect effect, benefits the decision maker by discouraging the bad sender from trying to persuade. It also reduces the sender's expected payoff because it increases the length of time for acceptance. With respect to the effects of the cost of communication for the sender, on top of some intuitive results, a decrease in it also lengthens the expected time of acceptance.

[1] Hence even if we endow the sender a set of cheap messages as available message, these only have the same role as silence.

While in the main analysis, we consider that the decision maker cannot make any form of commitment, we also characterize her optimal commitment problem. First, we show that the optimal commitment mechanism takes a stochastic form, in which the decision maker attaches the highest probability of acceptance to each node that prevents the bad type sender from trying to persuade. Furthermore, it can be shown that this does not harm the sender relative to the (best) equilibrium, which means that the optimal stochastic commitment can be a Pareto improvement. This is because the commitment makes it possible to avoid the case in which the bad sender tries to but fails to persuade the decision maker. In this case, which necessarily happens with a positive probability in the equilibrium, both pay wasteful communication costs.

In order to consider the case that it is hard to make stochastic commitment, we also examine a limited commitment method in which the decision maker can make only the non-stochastic commitment of requiring a predetermined amount of evidence. We show that even this limited commitment is beneficial for the decision maker relative to the best equilibrium when the sender's communication cost is low. However, interestingly, playing the best equilibrium is better when the sender's cost is high. This result comes from the fact that the equilibrium of the game may make the sender pay more communication cost ex-post than the gain from persuasion, which allows the decision maker to extract more information from him. In contrast, in the non-stochastic commitment, the sender is perfectly knowledgeable about the outcome of the persuasion at the beginning, and hence it is impossible to make him show a large number of evidence.

1.1 Related Literature

Our model is most closely related to the literature on strategic communication with verifiable messages, which is also called persuasion games. The most important benchmark was developed by Grossman (1981) and Milgrom (1981). They study a persuasion model in which the sender is not required to tell the truth in a precise manner, and show that we have complete unravelling of information. Shin (1994) studies a persuasion game in which the decision maker does not know how precise the sender's information is, and shows that unravelling of information breaks down. Verrechia (1983) incorporates a cost of information transmission for the sender to those models, and also shows that it prevents complete unravelling of information.[2] In the current, complete unravelling of information does not happen because the decision maker is not willing to pay the cost of communication up to the point that full information is obtained.[3] Forges and Koessler (2008) characterize the sets of equilibrium payoffs achievable with unmeditated communication in persuasion games with multi stages. Hörner and Skrzypacz (2011) study a dynamic model of verifiable information transmission in which a seller can transmit information gradually as the buyer makes payment for it.

In using a setting in which the sender gets a collection of binary signals about the state, this paper is related to Dziuda (2007) and Quement (2010). Dziuda (2007) offers a model in which a sender tries to persuade the decision maker to make a particular action by revealing verifiable

[2] Kartik, Ottaviani, and Squintani (2007) and Kartik (2009) study a model in which the sender's information is not verifiable but he bears a cost of lying and hence information is costly to falsify.

[3] Che and Kartik (2009) build a model of verifiable information but the sender has to pay the cost of information acquisition. They analyzes the problem of who to ask for advice, given the fact that full information revelation from the sender does not happen.

information. In her model, the persuader may be either a strategic agent or a truth teller. Quement (2010) constructs a model in which the sender has either a small number of evidence or a large number of evidence. Their question is if the strategic sender has an incentive to reveal unfavorable signals or not, and they show that the sender may do so. Although our model does not pay a particular attention for the question of whether the sender communicates unfavorable signals or not, it also has an equilibrium in which the sender communicates unfavorable signals (and it is proved to be inefficient). This is because even sending an unfavorable signal incurs the cost and thus it signals that the sender is a good type, who has confidence of being able to persuade in the end.

There are some studies that investigate a problem of persuasion as a mechanism design problem. In Glazer and Rubinstein (2004), the decision maker is allowed to check one piece of evidence of the sender's proposal, and they study mechanisms that maximize the probability that the decision maker accepts the sender's request if and only if it is justified. Sher (2010) generalizes the Glazer and Rubinstein's model in a way that both static and dynamic persuasion can be considered, and characterizes the relation between them. Kamenica and Gentzkow (2010) demonstrate that a sender can induce his favorite action from the decision maker by ingeniously designing the signal structure by which they can make Bayesian updating of information. This paper also addresses a similar problem of how the decision maker should design her acceptance rule by examining the optimal commitment problem.

Analytically, this study is closely related to a variant of the games of attrition where players use mixed strategies to resolve the dynamic strategic tension.[4] Hendricks and Wilson (1988) study a war of attrition in a complete information model. Kreps and Wilson (1982), and Ordover and Rubinsten (1986) consider models of attrition with asymmetric information. Although they build models on zero-sum payoff structure while we do not, their models are analytically similar to our model in the sense that some of players must play mixed strategies to have a gradual revelation of types in an equilibrium. An important difference is that in their studies, one of the informed players has a dominant strategy for the duration of the game, and as a consequence, all nodes of game are reached with a positive probability. However, in our study, no player has a dominant strategy and all players' incentives are endogenously determined in the game. One more important difference is that in the variant of war of attrition, duration works as a indirect signal about player's private information, which can be cost of fighting, cost of failing the agreement, time preference, and so on, through showing how much they can "burn money". In our model, in contrast, private information is gradually revealed by the process of the decision maker directly asking the sender. Baliga and Ely (2010) consider a model in which a principal uses torture to extract information from an informed agent. In equilibrium the informed agent reveals information gradually, initially resisting and facing torture but eventually he concedes.

This paper is also related to the literature on cheap-talk communication in dynamic models. Sobel (1985) develops a dynamic cheap talk model[5] in which the sender is either a friend or an enemy of the decision maker, and examines the problem of how long the sender should spend constructing his reputation and when he should deceive the decision maker. Aumann and Hart (2003), and Krishna and Morgan (2004) show that multiple exchanges of messages can convey

[4] A notable difference is that we formulate the game in discrete time rather than continuous time that is standard in game of attrition.

[5] For a benchmark model of cheap talk game, see Crawford and Sobel (1982).

more information than a single massage. Eso and Fong (2008) study a model with multiple senders where the decision maker can choose when to make her decision. They show that the threat of costly delay can induce instantaneous full revelation of information.

This paper is organized as follows. Section 2 introduces the basic structure of the model. In section 3, we provide analysis on the simplest example of the model. In Section 4, we provide general cauterization of equilibrium. Section 5, we do comparative static analysis. In section 6, we examine commitment problems. Proof of the theorems can be found in the Appendix.

2 Model

There are two players: a sender (persuader) and a decision maker, or DM hereafter.[6] The sender has a proposal that he would like the DM to accept. The quality of the sender's proposal θ (the state) is either 1 or -1; there is a common prior over the state. The sender does not observe the state θ[7] but he recieves $N \in \mathbb{Z}$ pieces of evidence that contains information about the quality of the proposal. Each piece of evidence is either good (G) or bad (B). The vector of evidence $e \in \{G, B\}^N$ is drawn from a distribution $g(e|\theta)$. Given the environment, we also have the probability distribution over θ conditional on the realization of e. We assume that the pieces of evidence are interchangeable in the sense that the $\mathbb{E}[\theta|e]$ depends only on the number of pieces of good evidence in e. Given the assumption, we have the expected value of θ conditional on the realization of j pieces of good evidence among N, and denote it by $\mathbb{E}[\theta|j]$. We assume that $\mathbb{E}[\theta|j]$ is increasing with j, which means that more good evidence makes the prospect of the proposal better. In order to exclude trivial cases, we assume that $\mathbb{E}[\theta|0] < 0$ and $\mathbb{E}[\theta|N] > 0$.

Denote by ξ the threshold number of good pieces of evidence that makes the expected value of θ strictly higher than zero, that is,

$$\mathbb{E}[\theta|\xi - 1] < 0 \leq \mathbb{E}[\theta|\xi].$$

Furthermore, denote by f the unconditional density over the realization of the number of good pieces of evidence; the sender's *type*. The DM wants to accept the sender's proposal if $\theta = 1$ and reject if $\theta = -1$, and hence she cares about the sender's type. Everything except the realization of the sender's type is common knowledge.

To illustrate our setting, as an example, think of the following simple scenario which is taken from the literature on strategic voting.[8] The prior probability that the state being 1 is 1/2. There are two pieces of evidence, i.e., $N = 2$. Each piece of evidence is independent with each other. Conditional on $\theta = 1$, the probability that a realization of a piece of evidence is G is $p > 1/2$, and conditional on $\theta = -1$, it is $1 - p$. Then it follows that

$$\mathbb{E}[\theta|0] = \frac{1-2p}{p^2 + (1-p)^2}, \ \mathbb{E}[\theta|1] = 0, \text{ and } \mathbb{E}[\theta|2] = \frac{2p-1}{p^2 + (1-p)^2},$$

Hence $\mathbb{E}[\theta|2] > \mathbb{E}[\theta|1] > \mathbb{E}[\theta|0]$ and ξ is one. Also, $f(0) = f(2) = \frac{1}{2}\{p^2 + (1-p)^2\}$ and $f(1) = 1 - p^2 - (1-p)^2$. Our setting allows more general cases relative to this example in a

[6] Throughout, we use female pronouns for the decision maker and male pronouns for the sender.
[7] In our model, it actually does not matter at all whether we assume that the sender observes θ or not.
[8] See Feddersen and Pesendorfer (1998) for an example.

sense that we do not necessarily assume that each piece of evidence is independent with each other.

2.1 Dynamic Game of Persuasion

In the process of the game, the decision maker's turn and the sender's turn alternate. At each turn of the DM, she has three choices: whether she accepts, rejects, or continues, which is interpreted as requiring a piece of evidence from the sender. At each turn of the sender, he has three choices: communicating the DM about a good piece of evidence, a bad piece of evidence, or being silent. It is assumed that the sender cannot reveal more than one piece of evidence at a time, which is understood to be a technological constraint of communication. We can also think that it is extremely costly to communicate multiple evidence at a time. The game is terminated once the DM chooses to accept or reject.

The formal description of the model is as follows. Time is discrete and extends from 0 to ∞ that is denoted by $t \in T = \{0, 1, 2, .., \infty\}$. Before everything starts, Nature draws $\theta \in \{-1, 1\}$ and conditional on the realization, it chooses the sender type, the number of pieces of good evidence the sender has. The number j is the sender's private knowledge. In our model, it is assumed that the sender is not informed about the realization of θ, although it does not matter at all for the analysis. At period 0, the decision maker chooses one from $\{A, R, C\}$, where A, R, and C correspond to accept, reject, and continue (require a piece of evidence), respectively. If C is chosen, the game proceeds to period 1. In period 1, first the sender chooses $m_1 \in \{G, B, S\}$ under the condition that he can choose G (B) only when $j \geq 1$ $(j \leq N-1)$. Here, G and B mean to show a good or bad piece of evidence, respectively, and S means that the sender remains silent.[9] He can show a good (bad) piece of evidence only when he has at least one of it. Then the communication takes place. Then the DM chooses one from $\{A, R, C\}$ and in the case that C is chosen, the game proceeds to period 2. Now, in the beginning of period 2, the sender chooses $m_2 \in \{G, B, S\}$ under the condition that m_2 can be G only when $j \geq 2$ if $m_1 = G$ and $j \geq 1$ if $m_1 \neq G$. We have the symmetric condition for B as well. The rest of the game is described in the similar manner. The game terminates once the DM chooses either A or R.

Message history at period t is a sequence of messages communicated up to period t, and it is denoted with superscript by m^t. The set of all histories at period t is $M^t = \times_t \{G, B, S\}$, and the set of all histories is $M = \cup M^t$. Then define function $N_G : M \to \{1, 2, .., N\}$, $N_B : M \to \{1, 2, .., N\}$ and $N_S : M \to \{1, 2, .., N\}$ as the number of G, B, and S along message history m^t, respectively[10]. Obviously, we have $N_G(m^t) + N_B(m^t) + N_S(m^t) = t$. In the following analysis, the set of available message for type j sender after message history m^t is denoted by $M(m^t, j)$, that is $S \subset M(m^t, j)$ for all (m^t, j) and

$$G \in M\left(m^t, j\right) \text{ iff } j > N_G\left(m^t\right) \text{ and } B \in M\left(m^t, j\right) \text{ iff } N - j > N_B\left(m^t\right).$$

Therefore, $M(m^t, j)$ cannot contain G (B) if the sender runs out of good (bad) piece of evidence to communicate on the history m^t.

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[9] We can also change the model by allowing sender to send a cheap message from a finite set of cheap massages, without adding any change to the results.

[10] More precisely, $N_G(m^t) = |\{k|m_k = G,\ k \leq t\}|$, $N_B(m^t) = |\{k|m_k = B,\ k \leq t\}|$, and $N_S(m^t) = |\{k|m_k = S,\ k \leq t\}|$.

In our model, persuasion is costly for both players. We can simply think that it is costly because it takes valuable time and there are cognitive costs that they have to pay to make the DM understand the sender's explanation. Specifically, we want to think that communication is costly for the DM because it delays his decision making, and it is costly for the sender because formulating or explaining evidence to the DM is costly due to cognitive costs. In this sense, we will use the term "communication cost" in a broad sense including delay cost.

We can also take the interpretation of Dewatripont and Tirole (2005)'s observation, which states that information is neither hard nor soft initially, but the degree of softness is endogenously changed. Only by combining two sides' mutual effort can they turn the information into hard. If we take this interpretation, we assume that the degree of softness is zero-one[11]. To make things simple, we simply assume that the cost of communicating a piece of evidence is fixed for both sides. Thus the communication technology for our model is specified as follows:

Communication cost for the DM.

The (one time) cost of communication for the DM is represented by a function $\eta : \{G, B, S\} \to \mathbb{R}$, where

$$1 > \eta(G) = \eta(B) > 0 \text{ and } \eta(S) > 0.$$

Communication cost for the sender.

The (one time) cost of communication for the sender is represented by a function $\delta : \{G, B, S\} \to \mathbb{R}$, where

$$\delta(G) = \delta(B) > 0 \text{ and } \delta(S) = 0.$$

These say that communicating a piece of evidence G or B is costly for the DM as well as the sender, and in particular, communicating a piece of good evidence incurs strictly positive cost for the both. Silence is also costly for the DM,[12] while it is not for the sender. Although it is possible to work on a model of positive silence cost for the sender, the assumption simplifies some of the mathematical expressions that appear later. Assumptions of $\eta(G) = \eta(B)$ and $\delta(G) = \delta(B)$ are purely for notational simplicity, and it is straightforward to extend the model by relaxing those assumptions.

In the following, we simply denote $\eta(G)$ (and hence also $\eta(B)$) by η, $\eta(S)$ by η_S, and $\delta(G)$ (and hence also $\delta(B)$) by δ. The communication costs that two players have to pay depend on how many times pieces of evidence or silence are communicated multiplied by the communication cost. To shorten the notation, we define the functions that represent the costs of communication along the message history m^t as follows:

$$C_{DM}\left(m^t\right) = \eta\{N_G\left(m^t\right) + N_B\left(m^t\right)\} + \eta_S N_S\left(m^t\right)$$

for the decision maker and

$$C_S\left(m^t\right) = \delta\{N_G\left(m^t\right) + N_B\left(m^t\right)\}.$$

for the sender. As soon as the DM takes an action both of the players get their respective payoffs. The DM's (expected) payoff when the seeder type is j, which is denoted by

[11] In Dewatripont and Tirole (2005), in contrast, the level of effort, which can increase the probability of being able to make information hard, is chosen by both sides. They examine the problem of moral hazard in team in that setting.

[12] It is also possible to choose a model setting in which silence does not incur cost for the DM. We chose the current setting because it generates inessential multiplicity of equilibrium.

$U_{DM}(a, j, m^t)$, depends on the particular action (accept or reject) taken by the DM, the type of the sender, and the communication history after which the DM takes action:

$$U_{DM}\left(A, j, m^t\right) = \mathbb{E}[\theta|j] - C_{DM}\left(m^t\right) \text{ and } U_{DM}\left(R, j, m^t\right) = -C_{DM}\left(m^t\right).$$

When the DM accepts the proposal, her payoff depends on the sender type through the term $\mathbb{E}[\theta|j]$, which should be interpreted that the actual payoff of the decision maker is θ and its expected value is taken.[13] If the DM rejects the proposal, she has an outside option that ensures her payoff of zero, and just pays her communication cost.

The sender's payoff, which is denoted by $U_S(A, m^t)$, depends only on the particular action taken by the DM and the communication history after which the DM takes the action:

$$U_S\left(A, m^t\right) = V - C_S\left(m^t\right) \text{ and } U_S\left(R, m^t\right) = -C_S\left(m^t\right),$$

where $V \geq \delta$. Hence the sender's payoff is V, which is the gain from persuading the DM, minus the communication cost if the DM accepts his proposal. It implies that the sooner he can persuade the DM, the higher his payoff is. It is possible that even if he could eventually persuade the DM, the communication cost is larger than the gain of persuasion V. On the other hand, he just pays the communication cost when the DM ends up with rejecting the proposal.

Hence in our model, the cost of communication, which can be interpreted as a time cost, appears in the players' payoffs in an additively separable form. An alternative setting is one in which players' payoffs are discounted as time goes by. This setting, however, cannot generate the equilibrium that we will characterize; in such a setting the sender does not have an incentive to give up persuasion because his payoff just shrinks and never becomes negative. On the other hand, it is possible to model the DM's payoff in a discounted form and still get the same type of equilibrium, because even in such a setting, she faces the same trade-off between prompt decision making and information collection.

Now we define the strategies of two players. The sender's (behavior) strategy is a probability measure $\alpha(\cdot, m^t, j)$ over available messages $M(m^t, j)$, parameterized by (m^t, j). It represents the type j sender's strategy after message history m^t, and $\alpha(m, m^t, j)$ is the probability that he chooses a particular message $m \in \{G, B, S\}$. The strategy of period 1 is denoted by $\alpha(\cdot, \varnothing, j)$, by using a convention of notation $m^0 = \varnothing$. On the other hand, the DM's (behavior) strategy is a probability measures β over $\{A, R, C\}$, parameterized by m^t. Her strategy at period 0 is $\beta(\cdot, \varnothing)$.

We introduce notations and definitions to be used in the subsequent analysis. As the game proceeds, the DM's belief about the sender type evolves. Her belief, which is parametrized by message history m^t is represented by a vector of function $B_n : M \to [0, 1]$ for $n = 0, 1, 2, .., N$ such that $\sum_{n=0}^{N} B_n(m^t) = 1$, that is, $B_j(m^t)$ is the probability that the DM attaches to the event that the sender type being j, after communication history m^t.

Given a sender's strategy α, we can define the probability that a particular message history is followed, that is

$$\varphi\left(m^t\right) = \sum_{j=0}^{N} f(j) \Pi_{\tau=1}^{t} \alpha\left(m_\tau, m^{\tau-1}, j\right).$$

[13] More precisely, the DM's utility depends on the state, action, and message history that is written as $U_{DM}(A, \theta, m^t) = \theta - C_{DM}(m^t)$ and $U_{DM}(R, \theta, m^t) = -C_{DM}(m^t)$.

Given the DM as well as the sender's strategy, we can define the set of message history that can be reached with strictly positive probability

$$\Delta = \{m^t \mid \sum_{j=0}^{N} f(j)\, \Pi_{s=1}^{t} \alpha\left(m_s, m^{s-1}, j\right) \cdot \beta\left(C, m^{s-1}\right) > 0\}.$$

We simply call elements of Δ on-equilibrium message history.

In the following analysis, we use the following notations for the ease. The notation (m^t, m) reads "a message history such that m^t is followed by m". In particular, $(m^t, G) \in M^{t+1}$ represents message history m^t followed by G (Also, $(m^t, B) \in M^{t+1}$ should be read similarly). Also, we denote by $G^t \in M^t$ the message history at period t that contains only G.

2.2 Equilibrium

Our solution concept is that of perfect Bayesian equilibrium, as is defined in Fudenberg and Tirole (1991, Definition 8.2).[14] This requires that after each history of messages $m^t \in M$, the DM maximizes her expected payoff given her belief about sender's type and her and the sender's future behavior, and also the sender maximizes his expected payoff given the DM's strategy.

In order to formally define the equilibrium, we first define the value function of the players. In our game, the decision of each period necessarily depends upon the decisions of the next period, and that in turn depends on the decision of the following period, and so on. The value function we will define makes it possible to summarize all the information about the future play of the game that is necessary for making the current decision.

We start by defining the value function for the DM. In order to do this, let $\varphi(|m^t)$ be a probability distribution function over $\{G, B, S\}$, parametrized by $m^t \in M$, which can be interpreted as the DM's belief about next period's messages she will hear from the sender, should she continue. We say that a function $V_{DM} : M \to \mathbb{R}$ is a value function for the DM given (φ, B) if, for all $m^t \in M$,

$$V_{DM}\left(m^t\right) = \max\{\max_{a \in \{A,R\}} \sum_{j=0}^{N} B_j\left(m^t\right) U_{DM}(a, j, m^t), \sum_{m \in \{G,B,S\}} \varphi(m|m^t) V_{DM}\left(m_t, m\right)\} \tag{1}$$

and

$$\lim_{t \to \infty} V_{DM}(m^t) = -\infty \text{ for all } \{m_t\}_{t=0}^{\infty}. \tag{2}$$

The left element in the right hand side of (1) is the expected utility for the DM when she makes decision immediately after message history m^t, whereas the right element is the expected value for waiting for one more period. The value of communication history m^t is determined by their bigger one. The next condition (2) is understood to be the counterpart of "no-ponzi game condition" in dynamic optimization problems in our model. In a typical formulation of a consumer's dynamic optimization problem, the no-ponzi game condition ensures that the consumer cannot keep borrowing money over time and accumulating debt and thereby makes his utility arbitrary

[14] Their definition is for finite multistage games. Here instead, the game has infinite stages and hence the definition follows is a slight generalization of it.

large. Condition (2) is the reminiscent of that restriction in our model, which is necessary to pin down the value function for the DM; without it, the uniqueness of the value function is not ensured. Note that (2) is the same as requiring $\lim_{t\to\infty} V_{DM}(m^t) = -C_{DM}(m^t)$, because silence is costly for the DM and hence $\lim_{t\to\infty} C_{DM}(m^t) \to \infty$ for all sequence of history $\{m^t\}_{t=1}^{\infty}$. We have the following lemma, which states that once we are given the sender's strategy and the DM's belief, the value function for the DM is uniquely determined.

Lemma 1 *Given* (φ, B), V_{DM} *is uniquely determined.*

Similarly, we can define the value function for the sender. In contrary to the value function for the DM, sender's value function should be parametrized by his type. We say that a function $V_S : M \times N \to \mathbb{R}$ is a value function for the sender type j given the DM's strategy β if

$$\begin{aligned} V_S\left(m^t, j\right) &= \beta\left(A, m^t\right) U_S\left(A, m^t\right) + \beta\left(R, m^t\right) U_S\left(R, m^t\right) \\ &\quad + \beta\left(C, m^t\right) \max_{m \in M(m^t, j)} V_S\left(\left(m^t, m\right), j\right) \end{aligned} \tag{3}$$

and

$$\lim_{t\to\infty} V_S\left(m^t, j\right) = -\lim_{t\to\infty} \delta\{N_G\left(m^t\right) + N_B\left(m^t\right)\} \text{ for all } \{m^t\}_{t=0}^{\infty}. \tag{4}$$

The max operator in the right hand side of (3) subsumes the fact that the sender behaves optimally at the next period. We also have no-ponzi game condition for the sender as well. Then we have the same lemma as when we defined the value function of the DM.

Lemma 2 *Given* β, V_S *is uniquely determined.*

Corollary 1 *Value function* V_{DM} *satisfies* $-C_{DM}(m^t) \leq V_{DM}(m^t) \leq 1 - C_{DM}(m^t)$ *for all* m^t *and value function* V_S *satisfies* $-C_S(m^t) \leq V_S(m^t, j) \leq V - C_S(m^t)$.

With above preparations, we can define the equilibrium. Think of the following conditions for a pair of strategies and the DM's belief $(\alpha, \beta, B, \varphi)$.

D1. The optimality of the sender's strategy at every history of messages:

$$\alpha\left(m, m^t, j\right) > 0 \text{ only when } m \in \arg\max_{m \in M(m^t, j)} V_S\left(\left(m^t, a\right), j\right).$$

D2. The optimality of the DM's strategy at every history of messages:

$$\begin{aligned} \beta\left(C, m^t\right) &> 0 \text{ only when } V_{DM}(m_t) = \mathbb{E}[V_{DM}\left(m^{t+1}\right)|m^t)], \\ \text{and } a &\in \{A, R\},\ \beta\left(a, m^t\right) > 0 \text{ only when } V_{DM}(m_t) = \mathbb{E}[U(a, j, t)|m^t]. \end{aligned}$$

D3. Bayes' rule for the belief of the DM (B, φ) : For all $m^t \in M$,

$$\varphi\left(m_{t+1}|m^t\right) = \sum_{n=0}^{N} B_n\left(m^t\right) \alpha\left(m_{t+1}, m^t, j\right),$$

and if there is some j such that $\alpha(m_t, j, m^{t-1}) > 0$ and $B_j(m^{t-1}) > 0$,

$$B_j\left(m^t\right) = \frac{B_j(m^{t-1})\, \alpha(m_t, m^{t-1}, j)}{\sum_{n=0}^{N} B_n(m^{t-1})\, \alpha(m_t, m^{t-1}, n)} \text{ and } B_j\left(m^1\right) = \frac{f(j)\, \alpha(m_1, \varnothing, j)}{\sum_{n=0}^{N} f(n)\, \alpha(m_1, \varnothing, n)}.$$

$$B_j\left(m^t\right) = 0 \text{ for all } j < N_G\left(m^t\right) \text{ and } B_j\left(m^t\right) = 0 \text{ for all } j > N - N_B\left(m^t\right).$$

Our equilibrium is defined by those three conditions.

Definition 1 *A pair* $(\alpha, \beta, B, \varphi)$ *is a perfect Bayesian equilibrium iff it satisfies D1-D3.*

The first condition D-1 requires that the each time the sender chooses what to communicate, he chooses the one that maximizes his value. Note that this must hold not only for message histories that are reached with strictly positive probability (on-equilibrium history) but also the histories that are not supposed to reach with positive probability (off-equilibrium history). D-2 requires that same kind of behavior for the DM. She chooses to continue only when it maximizes her value, in which case her value $V_{DM}(m_t)$ is equal to $\mathbb{E}[V_{DM}(m^{t+1})|m^t)]$, and same for the choices of accept and reject.

Note that D-3 is stronger than simply using Bayes' rule in the usual fashion, since it applies to updating from period t to period $t+1$ when messages history m^t has probability zero, i.e., $m^t \notin \Delta$. The motivation for this requirement is that if $B_j(m^t)$ represents the DM's beliefs given m^t, and players follow their strategies at $t+1$, the DM should use Bayes' rule to form his belief in period $t+1$.[15]

The final requirement in D-3 simply says that the DM assigns zero probability to the type of the sender who has strictly smaller number of pieces of good (or bad) evidence than already shown. In the terminology of incomplete information game, the set

$$\{(j, m^t) \,|j \geq N_G(m^t) \text{ and } N-j \geq N_B(m^t)\} \subset \{0, 1, .., N\} \times M$$

is the information set for the DM after getting message m^t, and hence the DM has to put all the probability mass in this set.

Note how the two belief functions B and φ play different roles in the DM's decision making. The belief function B, which shows the DM's belief over how good the proposal is, is relevant for choosing whether to accept or reject, if she has to make decision immediately. On the other hand, the belief function φ, which shows the DM's beliefs about the sender's behavior at the next period, is relevant for choosing whether to decide immediately or to continue.

We conclude this section by showing some immediate results that follow almost directly from the definition of the equilibrium. The first one says that once the sender communicates sufficient number of pieces of good evidence, the DM accepts the proposal for sure, and the sender just remains silent afterwards (thus such a node should be off-equilibrium). The proof is straightforward and hence omitted:

Claim 1 *1. In any equilibrium, for all* m^t *such that* $N_G(m^t) \geq \xi$, $\beta(A, m^t) = 1$.
2. In any equilibrium, for all m^t *such that* $N_G(m^t) \geq \xi$, $\alpha(S, m^t, j) = 1$ *for all* j.

Since the DM, after verifying that the sender's proposal has enough number of good pieces of evidence, already knows that her optimal action is to accept the proposal irrespective of the realizations of the rest of evidence, she does not pay more communication cost and reveal the rest of evidence. On the other hand, knowing that the DM will accept the proposal, the sender does not communicate remaining evidence by incurring the communication cost.

Given an equilibrium, let Ξ be the set of on-equilibrium history of termination with acceptance, that is, $m^\tau \in \Xi$ if and only if $\beta(A, m^\tau) = 1$ and $m^\tau \in \Delta$. From the definition, $\beta(A, m^s) < 1$ for all m^s that is a sub-history of m^τ. The next result says that the DM accepts the proposal for sure on equilibrium only after she is shown a piece of good evidence.

[15] For more discussion about the requirements, see Fudenberg and Tirole (1991).

Proposition 1 *For all $m^{\tau} = (m^{\tau-1}, m_{\tau}) \in \Xi$, it must hold that $m_{\tau} = G$.*

This follows because if there is a message history such that $(m^{\tau-1}, B) \in \Xi$ or $(m^{\tau-1}, S) \in \Xi$, even the lowest type sender among all types who may follow $m^{\tau-1}$ can get accepted at period τ, which implies that there is no screening of a bad sender takes place at period τ. This contradicts the fact that the DM chooses continue after $m^{\tau-1}$.

Remark 1 *One may think that we should impose more restrictions on off-equilibrium belief than when we are working on the usual signaling games. This is because in our game, the DM's decision to take an action or continue crucial depends on her belief after the current period. In particular, when she decides to take some action and thereby terminates the game, that decision must be based on her off-equilibrium behavior of herself and the sender, and moreover, even their off-equilibrium behavior also depends on further off-equilibrium behavior. Therefore, one may want to use the concept of sequential equilibrium (Kreps and Wilson (1982)), rather than perfect Bayesian equilibrium just because it imposes more restrictions on off-equilibrium belief of the players. However, in our game it can be shown that those two equilibrium concepts coincide in a fundamental sense. In order to show this, let us define the usage of the term outcome equivalence. Let $O := M \times \{A, R\}$ be the set of pairs of message history and the DM's action. Then, the outcome of the game is a probability distribution over O. Then we say that that two different strategies pairs $(\alpha', \beta', \varphi', B')$ and $(\alpha'', \beta'', \varphi'', B'')$ are outcome equivalent if they induce the same outcome. Note that in such a case we have $\beta'(\cdot, m^t) = \beta''(\cdot, m^t)$ on every $m^t \in \Delta'$ and $\alpha'(m, j, m^t) = \alpha''(m \cdot j, m^t)$ for every m if $j \in P(m^t)$ and $m^t \in \Delta'$ such that $\beta'(C, m^t) > 0$, where $\Delta' \subset M$ is the set of nodes that can be reached with strictly positive probability (the set of on-equilibrium history in equilibrium $(\alpha', \beta', \varphi', B')$), and $\Delta'' \subset M$ is defined in the similar manner. It is easy to see that those imply $\Delta' = \Delta''$. We will call a PBE $(\alpha, \beta, \varphi, B)$ a sequential equilibrium if there is a sequence of totally mixed strategy $(\alpha^{\lambda}, \beta^{\lambda}, \varphi^{\lambda}, B^{\lambda})$, with $\lambda \in \mathbf{N}_{+}$ such that for each $m^t \in M$ and j, it converges to $(\alpha(m^t, j), \beta(m^t), \varphi(m^t), B(m^t)) \in R^{9+N}$. Then we have the following claim.*

Claim 2 *For every PBE, there is an outcome equivalent sequential equilibrium.*

3 An Example

This section is devoted to the analysis of the special case in which the number of pieces of evidence is two and every piece of evidence should be good for the expected value of the proposal becomes positive, that is, $N = \xi = 2$. Although this case is a special case, it is useful for getting the idea of the construction of the equilibrium and it provides fundamental properties that are shared with more general cases.

The first observation is that the DM's strategy of continuing until two pieces of good evidence is communicated is not supported as an equilibrium. This is because this naive strategy makes the type 1 sender give up persuasion by silence from the beginning, because he knows that two pieces of good evidence are necessary to persuade the DM. However, it makes the DM strictly better to accept immediately after one piece of good evidence, because it already screened out low type sender. This implies that equilibrium necessarily involves mixing strategies to resolve the tension.

To focus on the most interesting case, we impose the following assumptions.

$$\mathbb{E}[\theta|j \geq 1] \geq 0^{16} \text{ and } -\mathbb{E}[\theta|1]f(1) \geq \eta f(2) + \eta_S f(1). \tag{5}$$

The second condition says that the cost of communication is low enough, compared the loss from accepting the type 1 sender's proposal. Roughly speaking, the DM is willing to pay the communication cost if she can screen out type one sender when she knows that the sender is either type 1 or 2.

Even in this special case, we have a plethora of equilibria. The next proposition characterizes one of those, where the reason we focus on it is fully discussed in the next section (it is actually the *ex-ante* best equilibrium for the DM). In the statement of the theorem, we omit the description of off-equilibrium behaviors, because it is straightforward to specify those. Remember that the second element of the sender's strategy α is a message history, and the third element represents the type of the sender.

Proposition 2 *A pair of strategies that satisfies the followings is an equilibrium.*

1. Type 2 sender communicates good pieces of evidence in row: $\alpha(G, \varnothing, 2) = \alpha(G, G, 2) = 1$, *and type 0 sender chooses silence at period 1:* $\alpha(S, \varnothing, 0) = 1$.

2. Type 1 sender mixes at period 1:

$$\alpha(G, \varnothing, 1) = c \text{ and } \alpha(S, \varnothing, 1) = 1 - c, \text{ where } c = -\frac{f(2)\eta}{f(1)(\mathbb{E}[\theta|1] + \eta_S)}$$

3. At period 2, the DM accepts if she has been communicated two pieces of good evidence, and rejects otherwise:

$$\beta(A, (G, G)) = 1 \text{ and } \beta(R, m^2) = 1 \text{ if } m^2 \neq (G, G).$$

4. At period 1, the DM mixes between continuing and acceptance if she has been communicated a piece of good evidence, and rejects otherwise:

$$\beta(A, G) = \delta/V, \ \beta(C, G) = 1 - \delta/V, \text{ and } \beta(R, m_1) = 1 \text{ if } m_1 \neq G.$$

5. At period 0, the DM continues ($\beta(C, \varnothing) = 1$) *if* $W \geq \max\{0, \mathbb{E}[\theta]\}$, *rejects* ($\beta(R, \varnothing) = 1$) *if* $0 > \max\{W, \mathbb{E}[\theta]\}$, *and accepts* ($\beta(A, \varnothing) = 1$) *if* $\mathbb{E}[\theta] > \{W, 0\}$, *where*

$$W = cf(1)(\mathbb{E}[\theta|1] - \eta) + f(2)(\mathbb{E}[\theta|2] - \eta) - \{f(0) + (1 - \alpha) f(1)\}\eta_S.$$

The second period strategies are easy to see. The DM accepts the sender's proposal if the sender communicates the second piece of good evidence again and rejects otherwise, which induces the sender to communicate the last piece of good evidence if he still has it. For the first period strategies, after checking one piece of good evidence, the DM mixes between accepting and continuing. The probability that she accepts is δ/V, which makes the sender type 1 be indifferent between trying persuasion (by communicating good evidence) and giving up by being silent. On the other hand, the probability that type one sender tries persuasion is set in a way that the DM is indifferent between accepting and continuing and thereby screening it out at

[16] This is rewritten as $f(1)\mathbb{E}[\theta|1] + f(2)\mathbb{E}[\theta|2] \geq 0$, which is compatible with $\xi = 2$.

period 1. Note that if this probability is too low, at period one after checking one piece of good evidence, the DM is sure enough that the sender is type 2 and she strictly prefers to accept, and if it is too high she strictly prefers to continue. The expression c, which is the type one sender's trial probability $\alpha(G, \varnothing, 1)$, follows from the condition[17]

$$-\frac{cf(1)}{cf(1)+f(2)}\mathbb{E}[\theta|1] = \frac{cf(1)}{cf(1)+f(2)}\eta_S + \frac{f(2)}{cf(1)+f(2)}\eta.$$

The left hand side, the conditional probability that the sender is type 1 after communicating one good evidence is multiplied by the expected loss from acceptance, is the benefit of communicating one more time. The right hand side is the expected cost from communicating one more time, given the sender's strategy. Those two must be equal, because the DM must be indifferent between acceptance and communicating one more time.

At period zero, if the benefit from proceeding to period 1 is higher than the expected payoff from accept or reject without communication, the DM chooses continue. In such a case, we have W, which is characterized in the proposition, becomes $V_{DM}(\varnothing)$.

An important point to note is that at period one, it is the optimal for type 2 sender to communicate a piece of good evidence, because he is sure to be able to persuade the DM. This is so even when 2δ, which is the communication cost he ends up paying is larger than V. This implies that he is expecting "success with regret" to happen with some probability at the beginning of the game, because at period 1 after being required one more piece of evidence, his first communication cost is sunk and responding to the DM's request and showing the second good evidence becomes the optimal.

The specific equilibrium provided in Proposition 2 has some special characteristics that we focus in the next section. First, a piece of bad evidence is never communicated on-equilibrium. Second, the acceptance probability after communicating a good evidence is δ/V or 1. Finally, silence meets immediate rejection. Although there are other equilibria that do not satisfy those properties, we will discuss in the next section that equilibria that have those properties are more plausible relative to other equilibria.

An important note is that we can also have an equilibrium in which only one good evidence is needed to persuade the DM. Actually, if

$$f(1)(\mathbb{E}[\theta|1]-\eta) + f(2)(\mathbb{E}[\theta|2]-\eta) - f(0)\eta_S \geq \max\{0, \mathbb{E}[\theta]\},$$

the DM chooses to continue at time zero even if she knows that she is able to screen only type zero sender out. In such an equilibrium, both type one and two sender success persuasion by communicating only one piece of good evidence. Obviously, the equilibrium payoff is lower for

[17] Alternatively, we can write it as

$$\frac{f(2)}{cf(1)+f(2)}\mathbb{E}[\theta|2] + \frac{cf(1)}{cf(1)+f(2)}\mathbb{E}[\theta|1] = \frac{f(2)}{cf(1)+f(2)}(\mathbb{E}[\theta|2]-\eta) + \frac{cf(1)}{cf(1)+f(2)}\eta_S,$$

where the left hand side represents the DM's expected payoff from accepting the sender's proposal after checking a single piece of good evidence, while the right hand side is her expected payoff from continue and screen type one sender out.

the DM and higher for the sender, relative to the equilibrium of Proposition 1 (this fact is generalized in subsection 4.3).

In the rest of this section, we investigate more about the equilibrium in our example. First we do some comparative statics with respect to the parameter values η and δ. We start it by looking at the effect of δ.

Proposition 2 shows that the sender's cost δ has no effect on the DM's expected payoff. On the other hand, it has negative effect on the sender's expected payoff $\mathbb{E}_j[V_S(\varnothing, j)]$. To see this, think of the case that we increase δ. It has no effect on type 0 sender's payoff, because it does not participate in the persuasion process. Also, it has no effect on type 1 sender's expected payoff, because the increase of the cost is exactly offset by period 1's acceptance probability. Type 2 sender's expected payoff, however, will be decreased because period $2'$s acceptance probability is still one, and thus does not fully compensate the burden of the increase in the cost. It is easily seen that the increase in sender's cost decreases the expected time of DM's decision making through the increase the acceptance probability at period 1.

We next discuss the comparative statics with respect to the DM's cost of communication, η. Proposition 2 demonstrates that it actually has no effect on the equilibrium payoff of the sender, $\mathbb{E}_j[V_S(\varnothing, j)]$, as long as (5) is satisfied, because it does not affect the acceptance probability at period one and two.

It is obvious that η has a strictly negative relationship with $V_{DM}(\varnothing)$. An interesting fact is that when η decreases, the DM can enjoy not only direct effect as well as indirect effect of the decrease. It is seen by the following relation:

$$\frac{\partial V_{DM}(\varnothing)}{\partial \eta} = \underbrace{-\alpha(G, \varnothing, 1) f(1) - f(2)}_{\text{Direct effect } (-)} + \underbrace{\frac{\partial \alpha(G, \varnothing, 1)}{\partial \eta} f(1)(\mathbb{E}[\theta|1] - \eta + \eta_S)}_{\text{Indirect effect } (-)},$$

$$\text{where } \frac{\partial \alpha(G, \varnothing, 1)}{\partial \eta} = \frac{-f(2)}{f(1)(\mathbb{E}[\theta|1] + \eta_S)} > 0.$$

The direct effect is obvious. Since the sender will communicate a piece of good evidence with probability $\alpha(G, \varnothing, 1) f(1) + f(2)$ at period one, it becomes the first order effect on the decrease in η. Indirect effect stems from the fact that the DM must be indifferent between communicating and accepting after communicating once. To keep her indifferent after η gets smaller, the probability that type 1 sender tries to persuade should be suppressed so that the gain from screen that type out at period 2 gets smaller.

An important implication is that the DM wants to make commitment if she can write down a contingent plan to follow, rather than playing the original game. In fact, it is easy to see that the following method of commitment, if possible, makes the DM better off for sure: the DM accepts the offer with probability slightly smaller than δ/V at period one if a piece of good evidence is shown. At period two, she accepts for sure if the second piece of good evidence is shown. She rejects immediately when the sender shows something else. This commitment makes the DM better off because if the sender does not have two good pieces of evidence, he remains silent from the beginning and hence the commitment makes it possible to avoid accepting type one sender's proposal, that happens with some probability in the equilibrium of the original game.

We can also see that the DM can make herself better off even if she can make a limited form of commitment. Think of the commitment in the following method: the DM commits to check two pieces of evidence as long as the sender tries to communicate, and she accepts the proposal if two good pieces of evidence are shown. If, on the other hand, the sender chooses silence, the DM immediately rejects the proposal. To ensure that the sender's incentive compatibility is satisfied, we assume that $V \geq 2\delta$. Then the expected utility for the DM from this limited method of commitment is

$$V_C = f(2)\,\mathbb{E}[\theta|2] - 2f(2)\,\eta - (f(0) + f(1))\,\eta_S.$$

With probability $f(2)$ the sender is type 2 and the DM has to pay the communication cost of 2η. Otherwise, the sender is a bad type (type 0 or 1) and she will pay just a period cost of silence.

The expected payoff from this limited commitment is higher than W, which is the expected payoff from playing the original game. Actually, it is computed as

$$V_C = W + \alpha(G, \varnothing, 1)\, f(1)\,\eta, \tag{6}$$

Note that $\alpha(G, \varnothing, 1)$ is the probability that the decision maker can prevent type one sender from persuasion by making the commitment, relative to the equilibrium of the game. Given the equilibrium strategy of the sender, the following strategy is an optimal for the DM: continue after the first pieces of good evidence and accept after the second piece of good evidence, and otherwise reject. Then, the decision maker expects that if the sender is type 1, she communicates a piece of good evidence with probability $\alpha(G, \varnothing, 1)$ and silent with probability $1 - \alpha(G, \varnothing, 1)$. In the former case, she will end up being silent in the next period. Hence, the expected communication cost with sender type 1 in the equilibrium is $\eta_S + \alpha(G, \varnothing, 1)\,\eta$, while it is just η_S when she makes the commitment. Since the expected communication cost with sender type 2 is the same between the equilibrium and the commitment (2η), the relation (6) follows. In a nutshell, the commitment makes it possible to avoid checking type one sender's piece of good evidence, by completely discouraging it from persuasion. Whether the DM can make herself better off when the condition $V \geq 2\delta$ does not hold is discussed in section 6.

Although this simple example is enough to give the intuition to some of the important results that are valid for more general cases, there are still some questions that cannot be addressed by the simple example. For example, obviously, in the setting of $N = \xi$, we cannot have an equilibrium in which communicating a piece of *bad* evidence is on-equilibrium. However, such an equilibrium does exist in more general cases of $N > \xi$. To see this, think of the case in which ξ is large and all types of sender chooses B or S at period one. If the sender chooses G, the DM believes that the sender type is exactly 1 (off-equilibrium belief that we have no restriction), and hence immediately rejects the proposal. This in turn makes the sender avoiding G. Hence one possible question is if such an equilibrium is efficient or not, relative to other equilibria.

4 General Analysis

This section is for characterizing the properties of equilibria. In the first subsection, we give some basic properties of all equilibria. In the second subsection, we examine the properties that must be satisfied in an efficient equilibrium. Those are 1. a bad piece of evidence is never communicated. 2. Silence meets immediate rejection. 3. acceptance probability after good piece of evidence is δ/V or 1. In the third subsection, we characterize the best equilibrium for the DM, which is unique. We show that the best equilibrium must have the longest possible length of communication among Pareto optimal equilibria.

4.1 Properties of All Equilibria

As in most signaling games, our model also has a plethora of equilibria. However, it is possible to identify some important properties that all equilibria have to share.

The following theorem characterizes the most important properties of the equilibrium in our persuasion game. It says that every time a piece of good or bad evidence is communicated on-equilibrium, the DM must accept the proposal immediately with strictly positive probability. It also characterizes the lower bound of it.

Theorem 1 *In any equilibrium, if $m^{t+1} = (m^t, G) \in \Delta$ then $\beta(A, m^{t+1}) \in [\delta/V, 1]$. Also, if $m^{t+1} = (m^t, B) \in \Delta$ then $\beta(A, m^{t+1}) = \delta/V$.*

This result follows from the fact that communicating a piece of good or bad evidence incurs cost for the sender. If the probability of acceptance is very small right after (m^t, G), for the sender, communicating an evidence does not pay from myopic point, which implies that he expects acceptance with high probability in the future. It implies that every on-equilibrium history afterwards reaches a node that the DM accepts with some probability and hence acceptance is the best action for the DM at the node. However, it implies that acceptance is an optimal in all the contingencies, which implies that the DM should accept the proposal rather than continuing after (m^t, G).[18]

An important implication of the theorem is that in an equilibrium, the DM must not strictly prefer to continue each time she is communicated a piece of good or bad evidence. In constructing the equilibrium, this restriction imposes conditions about how much of bad sender types drop persuasion in the next period, and hence how much the DM's informational gain is. Note, however, that it is possible that the DM strictly prefers to continue at period 0, at the point where the sender has not yet paid the communication cost.

The next statement is an immediate corollary to Theorem 1, but also provides an important characterization of the equilibrium in our game. It says that silence has essentially no power of persuading the DM.

Theorem 2 *In an equilibrium, if $m^{t+1} = (m^t, S) \in \Delta$, then $\beta(A, m^{t+1}) = 0$.*

[18] In the alternative setting in which $\eta_S = 0$, the proposition can be rewritten as follows: if $m^{t+1} = (m^t, G) \in \Delta$ then there is a sequence of silence stage $m^{\tau}_{t+1} = (S, .., S)$ such that

$$\beta\left(A, m^{\tau}_{t+1}\right) = \sum_{s=t}^{\tau-1} \beta\left(A, \left(m^{s+1}, m_s\right)\right) \beta\left(C, \left(m^{s+1}, m_s\right)\right)^{s-t} \geq \delta/V,$$

that is, the DM must accept the proposal with probability higher than δ/V before they communicate another evidence.

From Theorem 1, every essential communication (not silent) meets immediate acceptance with a strictly positive probability. If moreover silence, which incurs no cost for the sender, also meets immediate acceptance with a strictly positive probability, acceptance is an optimal for all contingencies from the previous period's point of view. However, then for the DM it is strictly better to accept immediately at the previous period, which is a contradiction.

Theorem 1 implies that the value of a message history for the DM after a piece of evidence is shown (not silence) is equal to the expected payoff from accepting the proposal because it is an optimal action, where the expectation is taken with all the information she had gained through the message history. This is stated in the following corollary. Note that this should be the case even after a piece of bad evidence is communicated as long as it is on an on-equilibrium path.

Corollary 2 *In any equilibrium, it holds that*

$$V_{DM}\left(m^t\right) = \sum_{n=0}^{N} B_n\left(m^t\right) U_{DM}(A, n, m^t) \text{ for all } (m_{t-1}, m_t) \in M^{t-1} \times \{G, B\} \in \Delta.$$

4.2 Pareto Optimal Equilibria

In this subsection, we demonstrate that an efficient equilibrium is characterized by three properties.[19] Towards this end, first we define the set of Pareto optimal equilibria. Denote by $\mathcal{E}(\eta, \eta_S, \delta)$ the set of all equilibrium for a given pair of parameter values (η, η_S, δ). Also, we denote each value function with superscript e when we are mentioning it in a particular equilibrium e. We define the set of Pareto optimal equilibria as follows:

Definition 2 *Given (η, η_S, δ), the set of Pareto optimal equilibria $\mathcal{P}(\eta, \eta_S, \delta) \subset \mathcal{E}(\eta, \eta_S, \delta)$ is defined as follows: If $e \in \mathcal{P}(\eta, \eta_S, \delta)$, there is no $e' \in \mathcal{E}(\eta, \eta_S, \delta)$ such that $V_{DM}^{e'}(\varnothing) \geq V_{DM}^{e}(\varnothing)$ and $\mathbb{E}[V_S^{e'}(\varnothing, j)] \geq \mathbb{E}[V_S^{e}(\varnothing, j)]$, where the expectation is taken with respect to j[20], and one of the inequalities is strict.*

While we defined the set of Pareto optimal equilibria in a way that the sender's expected payoff is compared ex-ante, before the state of the world is realized, we can also define it in the interim way, in which the sender's expected payoff is compared after the state of the world is realized, i.e., the condition "$\mathbb{E}[V_S^{e'}(\varnothing, j)] \geq \mathbb{E}[V_S^{e}(\varnothing, j)]$" is replaced by "$V_S^{e'}(\varnothing, j) \geq V_S^{e}(\varnothing, j)$ for all j". However, all the results provided in this section are valid for whichever criteria we choose.

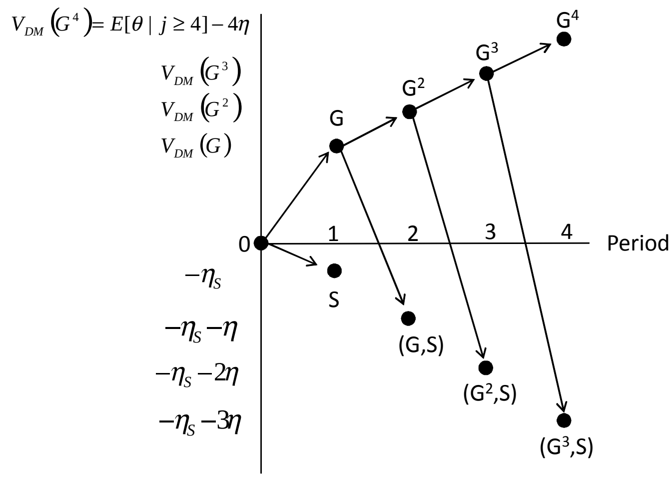
We define an important class of equilibrium that includes the set of Pareto optimal equilibria as a subset. In any equilibrium in the set, silence meets immediate rejection, even a single piece of bad evidence is never communicated, and acceptance probability is minimized among all possible ways of constructing an equilibrium.

[19] In this section, we ignore cases of some non-generic constellations of parameter values. More precisely, we exclude the cases in which

$$-\frac{f(j)\left(\mathbb{E}[\theta|j] + \eta_S\right)}{\sum_{k=j+1}^{N} f(k)} = \eta \text{ for some } j \leq \xi.$$

[20] Hence $\mathbb{E}[V_S^{e}(\varnothing, j)] = \sum_{j=0}^{N} f(j) V_S^{e}(\varnothing, j)$.

Figure 1.



Definition 3 *Given (η, η_S, δ), the set of benchmark strategy equilibria $\mathcal{B}(\eta, \eta_S, \delta) \subset \mathcal{E}(\eta, \eta_S, \delta)$ is defined as follows: If $e \in \mathcal{B}(\eta, \eta_S, \delta)$,*

1. A bad piece of evidence is never communicated, that is, $(m^t, B) \notin \Delta$ for all $m^t \in M$.

2. For all $(m^t, G) \in \Delta$, it holds that

$$\beta\left(A, \left(m^t, G\right)\right) \in \{\delta/V, 1\} \text{ and } \beta\left(C, \left(m^t, G\right)\right) = 1 - \beta\left(A, \left(m^t, G\right)\right). \tag{7}$$

3. For all $(m^t, S) \in \Delta$, it holds that $\beta(R, (m^t, S)) = 1$.

A property of benchmark strategy equilibrium is that once the DM chooses to enter the communication phase (period 1), all sender types higher than the number τ such that $\beta(A, G^\tau) = 1$ keep communicating good evidence until the proposal is accepted (it is the only optimal behavior given the DM's strategy). The DM never rejects the proposal from a high type sender, because he keeps communicating the good pieces of evidence, until the DM accepts eventually. Hence the DM does not make type I error in this sense.

Figure 1 describes how the value of the DM evolves over time in a benchmark strategy equilibrium. At period 1, the sender sends either G or S and the DM's value becomes $V_{DM}(G)$ and $-\eta_S$, respectively. Because in a benchmark strategy equilibrium only low type sender sends S at period 1, the DM's optimal action is to reject immediately and it results $V_{DM}(S) = -\eta_S$. Once the game reaches the node G, accepting the proposal is an optimal and she is indifferent between doing so and continuing. This means that $V_{DM}(G)$ is the appropriately weighted average of $V_{DM}(G^2)$ and $V_{DM}(G, S)$. The latter is $-\eta_S - \eta$ because again only low type sender sends S at period 2 and hence rejection is the optimal. At the final period where the DM accepts the proposal for sure, say period τ, her value reaches $\mathbb{E}[\theta | j \geq \tau] - \tau\eta$.

It is useful to define the "length" of persuasion for a benchmark strategy equilibrium. Given a benchmark strategy equilibrium $e \in \mathcal{B}(\eta, \eta_S, \delta)$, we call the number λ such that $\beta\left(A, G^\lambda\right) =$

1 but $\beta(A, G^t) = \delta/V$ for all $t < \lambda$, as the length of persuasion and denote it by $N_G(e)$. Actually, the length of persuasion is the number of pieces of good evidence to be required to make the DM accept for sure. Note, of course, that the DM may accept the proposal sooner with some probability and hence the terminology should be understood to be an abbreviation of "maximum possible length of persuasion".

An important property of benchmark strategy equilibrium follows directly from the definition.

Claim 3 *1. In a benchmark strategy equilibrium e, for all $j < N_G(e)$, $V_S(\varnothing, j) = 0$. Moreover, for all $j < N_G(e)$ and $j \le t < N_G(e)$, $V_S(G^t, j) = (t-1)\delta$.*

2. In a benchmark strategy equilibrium e, for all $j \ge N_G(e)$, $V_S(\varnothing, j) > 0$. Moreover, for all $j \ge N_G(e)$ and $t < N_G(e)$, $V_S(G^t, j) > (t-1)\delta$.

In a benchmark strategy equilibrium, after each message history, the sender's has only two choices; communicating a piece of good evidence, or being silent.[21] Because silence effectively implies giving up persuasion, the sender's strategy is characterized by a "dropping vector". Formally, type j sender's strategy is characterized by a ξ dimensional vector

$$d_j = (d_j^1, d_j^2, ..., d_j^\xi),$$

where d_j^n, which is $\alpha(S, G, G^{n-1})$, represents the probability that type j sender drops persuasion by silence at n's trial, i.e., if $d_j^1 = 1$, type j sender drops at period 1 for sure. Obviously, in a benchmark equilibrium e, for all type $j \ge N_G(e)$, $d_j^n = 0$ for all $n \le N_G(e)$, because it never drop out until eventually persuading the DM (this follows from Claim 3). We denote $N \times \xi$ dimensional vector $(d_1, d_2, ..., d_N)$ (collection of all sender types's strategy) by simply d in the subsequent analysis.

The next proposition shows the equations that characterize our benchmark strategy equilibrium.

Proposition 3 *Sender's strategy with dropping vector d such that $d_j^1 > 0$ for some j is supported as a benchmark strategy equilibrium if and only if there is κ such that*

$$\begin{aligned} & -\sum_{j\ge t}^{\xi} d_j^{t+1}\Pi_{s=1}^{t}(1-d_j^s) f(j)\,\mathbb{E}[\theta|j] \\ = & \;\eta \sum_{j\ge t+1}^{N} \Pi_{s=1}^{t+1}(1-d_j^s) f(j) + \eta_S \sum_{j\ge t}^{\xi} d_j^{t+1}\Pi_{s=1}^{t}(1-d_j^s) f(j) \end{aligned} \tag{8}$$

for all $t < \kappa$ and $\Pi_{s=1}^{\kappa+1} d_j^s = 0$ for all $j \ge \kappa + 1$, and

$$\sum_j^N \left(1 - d_j^1\right) f(j)\left(\mathbb{E}[\theta|j] - \eta\right) - \eta_S \sum_j^N d_j^1 f(j) \ge 0. \tag{9}$$

[21] In a benchmark strategy equilibrium, the acceptance probability after a piece of bad evidence is communicated (off-equilibrium) is set to be small and hence silence, which incurs no cost, is better for the sender.

In equation (8), the left hand side is the expected gain from screening out low type sender by continuing at period t. From Theorem 1, after message history G^t, the DM's optimal action is acceptance and thus acceptance is the status quo action. By continuing, with the probability

$$\frac{\sum_{j\geq t}^{N} d_j^{t+1}\Pi_{s=1}^{t}(1-d_j^s)f(j)}{\sum_{j\geq t}^{N}\Pi_{s=1}^{t}(1-d_j^s)f(j)}, \tag{10}$$

she can know that the sender is a low type and change her action to rejection (a high type sender never give up persuasion). On the other hand, this incurs the cost of communication. With probability

$$\frac{\sum_{j\geq t+1}^{N}\Pi_{s=1}^{t+1}(1-d_j^s)f(j)}{\sum_{j\geq t}^{N}\Pi_{s=1}^{t}(1-d_j^s)f(j)},$$

the sender is a high type to show next piece of good evidence with whom the DM has to pay the communication cost of η. On the other hand, with probability (10), the sender chooses silence and the DM has to pay the communication cost of η_S. Proposition 3 requires that those two values with adequately weighted are equal with each other.

Note that (8) implies that if the DM accepts for sure at period κ, we must have

$$-\Pi_{s=1}^{\kappa-1}d_{\kappa-1}^{s}f(\kappa-1)\left(\mathbb{E}[\theta|\kappa-1]+\eta_S\right)=\eta\sum_{j\geq\kappa}^{N}f(j), \tag{11}$$

because at period κ, only type $\kappa-1$ sender drops persuasion by being silent and $d_{\kappa-1}^{\kappa}=1$. Therefore, if $-\left(f(l)\,\mathbb{E}[\theta|l]+\eta_S\right)<\eta\sum_{k\geq l+1}^{N}f(k)$ for all $l\geq j$, we have no way to have an equilibrium with the maximum length of persuasion longer than $j+1$. The condition (9) ensures that after the first piece of good evidence is communicated, the DM's optimal action is A. If this condition is not satisfied, the DM does not accept the proposal, which contradicts Theorem 1.

We have a corollary of Theorem 1 that is used in the subsequent analysis. It determines the value of the DM's value function at the beginning of the game by a simple formula.

Corollary 3 *In a benchmark strategy equilibrium, it holds*

$$V_{DM}(\varnothing)=\max\{0,\mathbb{E}[\theta],\sum_{j=1}^{N}(1-d_j^1)f(j)\left(\mathbb{E}[\theta|1]-\eta\right)-\sum_{j=1}^{N}d_j^1 f(j)\,\eta_S\} \tag{12}$$

In the cases of $V_{DM}(\varnothing)=0$ and $V_{DM}(\varnothing)=\mathbb{E}[\theta]$, the DM just rejects and accepts the proposal without requiring a piece of evidence, respectively. When those are not the case, the DM proceeds to period 1, and hence her value is determined by the weighted average of payoffs between the case that the sender communicates a piece of good evidence, where her optimal action is acceptance, and the case that he chooses silent, where her optimal action is rejection.

Here we comment on the general procedure to find out an equilibrium. The easiest way to find an equilibrium is to determine the sender's strategy backward. First, we determine the final period at which the DM accepts the proposal for sure, say period κ. Second, let

$$d_j^{\tau}=0 \text{ for all } j\geq\kappa \text{ and } \tau\leq\kappa,$$

that is, the sender type higher than κ keep communicating good pieces of evidence for sure. It must be so in the equilibrium because for the sender type higher than κ showing a piece of good evidence has a strictly higher continuation value than choosing silent and getting rejected. Then we can determine $\Pi_{t=0}^{\kappa-1}\left(1-d_{\kappa-1}^{t}\right)$ by (11). The rest of the values of d should be chosen in a way that (8) as well as $d_{j}^{1}\geq 0$ for all j is satisfied. If there is no such a way of choosing d, we have no equilibrium with communication. Finally, we see if

$$\sum_{j=1}^{N}(1-d_{j}^{1})f\left(j\right)\left(\mathbb{E}[\theta|1]-\eta\right)-\sum_{j=1}^{N}d_{j}^{1}f\left(j\right)\eta_{S}\geq\{0,\mathbb{E}[\theta]\}$$

holds. If it does, we can support $\beta\left(C,\varnothing\right)=1$ and hence we have a benchmark strategy equilibrium with the sender's dropping vector d.

Remark 2 *It can be shown that when $\xi\leq 4$, for a generic constellation of parameters, the first and the third conditions of a benchmark strategy equilibrium implies that second.*

Now we state the main result of this subsection. We denote by $\mathcal{B}(\eta,\eta_{S},\delta)$ the set of benchmark strategy equilibrium. Then we have the following theorem, which demonstrates that every efficient equilibrium is a benchmark strategy equilibrium.

Theorem 3 *For all (η,η_{S},δ), $\mathcal{P}(\eta,\eta_{S},\delta)\subseteq\mathcal{B}(\eta,\eta_{S},\delta)$.*

Theorem 1 in Section 3 demonstrates that even if there is an equilibrium that involves the communication of a bad piece of evidence, the DM must accept it with probability δ/V, before she communicates the next evidence. This means that even from a piece of bad evidence, the DM is actually positively updating the sender type. The message that the set of Pareto equilibria involves no piece of bad evidence says that the way such an equilibrium screens the sender type is not efficient for both players.

The fact that playing an equilibrium with a period of communicating a piece of bad evidence does not benefit the sender can be easily seen. Because communicating a piece of bad evidence only meets with acceptance probability of δ/V, which is just enough to recover the communication cost, playing another equilibrium that skips such a period does not harm the sender (and it is possible to construct such an equilibrium). On the other hand, the fact that it does not benefit the DM is not as straightforward as one may think. To see this, think of an equilibrium such that a piece of bad evidence should be communicated. Then, the sender type N, who has only good pieces of evidence, has to drop at some period. This equilibrium makes it possible to make the right hand side of (8), the cost of communication, smaller at each period. Accordingly, this reduces the dropping from the low type sender at each period (the left hand side of (8)) or equivalently, increases the dropping at period one, which itself benefits the DM. The question is whether this gain outweighs the loss of giving up the best type sender, $f\left(N\right)\mathbb{E}[\theta|N]$; the answer turns out to be negative (see the Appendix).

It is rather easy to see that silence should meet immediate rejection in an efficient equilibrium. From Theorem 2, the sender cannot be accepted after silence, which means that having such a period does not make him better off, while even silence is costly for the DM. Those imply that given an equilibrium that has silence that does not meet immediate rejection, it is possible to construct another equilibrium that skips such a period, which Pareto dominates the original equilibrium.

To see the reason that the probability of acceptance immediately after a piece of good evidence should be exactly δ/V or 1 in an efficient equilibrium, suppose that communicating a piece of good evidence, say for the third time has acceptance probability strictly higher than δ/V, that is, $\beta(A,G) > \delta/V$. Then all the sender types who have more than three pieces of good evidence will communicate them at least three times. However, in such a case, we can make another equilibrium by making the DM accepts the proposal with probability one after communicating three pieces of good evidence. This is an equilibrium, since the sender as well as DM's strategy in the original equilibrium remains to be an optimum. Now the sender is strictly better off because he can persuade the DM sooner, without harming the DM. Hence the acceptance probability immediately after a piece of good evidence is either maximized or minimized among all possible ways of constructing an equilibrium.

4.3 The Best Equilibrium

In this subsection, we characterize the best equilibrium for the DM (we say just the best equilibrium hereafter), that is, the equilibrium such that $V_{DM}(\varnothing)$, the value of the DM at the initial period, is maximized. Because it is proved that the best equilibrium is unique for all parameter values, it pins down the equilibrium on which we can do comparative statics (section 5). Also, it gives the highest benchmark with which the DM's equilibrium payoff is compared when we examine commitment problem (section 6).

The result given in the previous subsection already demonstrated that the best equilibrium, which must be a Pareto optimal equilibrium, is one of benchmark strategy equilibrium. Therefore, in this section, we focus our analysis exclusively on the set of benchmark strategy equilibrium.

The first property of the best equilibrium for the DM is that it is actually unique.

Proposition 4 *$\mathcal{B}(\eta, \eta_S, \delta)$ has a unique maximizer of $V_{DM}(\varnothing)$.*[22]

An important characteristic of the best equilibrium for the DM is that it maximizes the length of persuasion among the set of Pareto efficient equilibria. Intuitively, increasing the amount of good evidence necessary for persuasion discourages bad senders from trying to persuade.

Theorem 4 *If equilibrium e^* is the best equilibrium for the DM, there is no equilibrium e such that $N_G(e) > N_G(e^*)$.*

Note that there are multiple equilibria even if we focus on the ones that maximize the length of persuasion. Also, note that the theorem does not state that an equilibrium has a higher expected payoff than another equilibrium if the former has longer length of persuasion.[23] It only says that if an equilibrium is the best equilibrium, it must have the maximum length of persuasion.

[22] We regard two equilibria that are outcome equivalent identical.

[23] This statement holds in the special case of $N = \xi = 2$, where the best equilibrium is unique.

To see this result in the simplest case, suppose that there are two equilibria, one with the length of persuasion of 1 and the other with the length of persuasion of 2. In the former equilibrium, both type 1 and 2 senders try to persuade, which implies that after checking a single piece of good evidence, the value of the proposal is $\mathbb{E}[\theta|j \geq 1]$ for the DM. On the other hand, in the latter equilibrium, type 1 sender does not try to persuade with probability one, which implies that after checking one piece of evidence the value of the proposal is higher than $\mathbb{E}[\theta|j \geq 1]$.[24] Hence the value of the decision maker at period zero is higher in the latter because it screens out more bad sender (type one sender) by the first piece of evidence.

Hence the procedure of finding the best equilibrium involves 1. find the maximum length of persuasion. 2. given the maximum length of persuasion, find the way that the sender gives up persuasion over time in a way that the low type's trial by showing an evidence at period 1 is suppressed the most. In other words, given the length of equilibrium, we have to connect different periods by equation (8) in a way that $\left|\sum_j^{\xi-1} d_j^1 f(j)\, \mathbb{E}[\theta|j]\right|$ is maximized.

We can get the basic idea of characterizing the best equilibrium by rewriting the condition (8) as

$$-\sum_{j\geq t}^{\xi} d_j^{t+1}\Pi_{s=1}^{t}(1-d_j^s) f(j)\left(\mathbb{E}[\theta|j]+\eta_S\right) = \eta \sum_{j\geq t+1}^{N} \Pi_{s=1}^{t+1}(1-d_j^s) f(j)\,. \tag{13}$$

As we saw in the previous subsection, we can construct an equilibrium backward. We first determine the last period of persuasion, say κ, and let $d_j^t = 0$ for all $t \leq \kappa$ and $j \geq \kappa$, i.e, the sender type higher than κ never drop persuasion. Then, we choose elements of d backward so that equation (13) is satisfied for all period. At each period t, given the value of the right hand side, there are multiple ways to assign the probability of dropping, $d_j^{t+1}\Pi_{s=1}^{t}\left(1-d_j^s\right)$, among different types to make the equality hold. An important observation is that because the absolute value of $\mathbb{E}[\theta|j]$ is decreasing with j as long as $j \leq \xi$, if we decrease $d_j^{t+1}\Pi_{s=1}^{t}\left(1-d_j^s\right)$ a bit for some j, say by $\Delta d_j^{t+1}\Pi_{s=1}^{t}\left(1-d_j^s\right)$, we need to increase it for i for more than $\Delta d_j^{t+1}\Pi_{s=1}^{t}\left(1-d_j^s\right)$ if $i > j$. This re-allocation of the dropping probability leads to higher value of the right hand side at period $t-1$, which leads lower dropping at period 1, i.e., lower $\left|\sum_j^{\xi-1} d_j^1 f(j)\, \mathbb{E}[\theta|j]\right|$. This observation implies that we should use more lower types to tie the consecutive period in the equality (13).

Although it is possible to state the general algorithm to construct the best equilibrium that is applicable to general cases, we can introduce an assumption that makes the characterization of the best equilibrium easier. Towards this end, let ζ be the highest $j < \xi$ such that $\mathbb{E}[\theta|j] + \eta_S < 0$. Roughly speaking, ζ is the sender type that the DM does not dare to pay the communication cost to screen it out. Obviously, for any equilibrium e, $N_G(e) < \zeta$.

Think of the function $\Gamma : \{0, 1, .., \zeta\} \to \mathbb{R}$ that is defined as

$$\Gamma(j) = \left|\frac{f(j)}{\sum_{k=j+1}^{N} f(k)}\left(\mathbb{E}[\theta|j]+\eta_S\right)\right|.$$

The assumption we want to impose is the function Γ being decreasing. The function $\Gamma(j)$ is made by multiplying the loss from accepting type j sender's proposal with the probability that

[24] Under the condition that the latter equilibrium exists, in order to support the former the decision maker has to expect that type 2 sender does not show the second piece of good evidence with probability one at period 2.

the sender's type being j relative to the probability that the sender type is strictly higher than j. Roughly speaking, high $\Gamma(j)$ implies the DM has strong incentive to screen out type j sender, after having already screened out lower types. A sufficient condition for $\Gamma(j)$ to be decreasing is that $|\mathbb{E}[\theta|j]|$ decreases fast enough to compensate for the change in the term $f(j) / \sum_{k=j+1}^{N} f(k)$, which is likely to be increasing. Of course, if $f(j) / \sum_{k=j+1}^{N} f(k)$ is increasing with j, the assumption is automatically satisfied.

To see that the assumption makes it easier to find the maximum length of persuasion, see condition (11). Under the condition, the maximum length of persuasion is determined by the largest κ such that $\eta < \Gamma(\kappa - 1)$. From the condition, we know that $\eta < \Gamma(l)$ for all $l < \kappa$ and this implies that we can find dropping vectors which can take only values less than one, in such a way that (8) is satisfied at each period. If the condition is not satisfied, the fact that j is the largest number satisfying $\eta < \Gamma(\kappa - 1)$ does not necessary imply that the maximum length of persuasion is κ. To see this, think of the case that the assumption of Γ decreasing is not satisfied and $\eta < \Gamma(\kappa - 1)$ but $\eta > \Gamma(\kappa - 2)$. To be an equilibrium with maximum length of persuasion of κ, we must have (11) holds in order to support period $\kappa - 1$'s behavior of the DM (mixing between accepting and continuing) and we also

$$-\sum_{j \geq \kappa-2}^{\kappa-1} d_j^{\kappa-1} \Pi_{s=1}^{\kappa-2} \left(1 - d_j^s\right) f(j) \left(\mathbb{E}[\theta|j] + \eta_S\right) = \eta \{\Pi_{s=1}^{\kappa-1} \left(1 - d_j^s\right) f(\kappa - 1) + \sum_{j \geq \kappa}^{N} f(j)\},$$

in order to support the DM's period $\kappa - 2$'s behavior. Here, note that only type $\kappa - 2$ or $\kappa - 1$ sender can drop at period $\kappa - 1$. However, if $\eta > \Gamma(\kappa - 2)$, it may not possible to choose d in such a way that above equality is satisfied, which implies that the maximum length of persuasion should be shorter than $\kappa - 2$.

In sum, under the condition of $\Gamma(j)$ being decreasing, the maximum length of persuasion is determined by j that satisfies

$$\Gamma(j) > \eta > \Gamma(j+1). \tag{14}$$

is satisfied.

The assumption also makes it easier to find out the optimal dropping vector. As we discussed above, in finding the best equilibrium, we should use more lower type sender's dropping to tie the consecutive period by the equality (13). In the equilibrium characterized in the theorem, we use only type j sender's dropping to make period j's equation (13). Apparently, type j is the lowest possible type to drop at period t in a benchmark strategy equilibrium. If Γ is decreasing, it is ensured that once we can make the equation (13) at period j satisfied by letting only type j sender drops at period j, it is also possible to make the equations hold at previous periods in the same way.

Think of the following procedure to find out a N dimensional vector $c = (c_0, c_1, c_2, .., c_N)$.

· Step 1. Let $c_\xi = c_{\xi+1} = .. = c_N = 1$, and $c_0 = 0$. Find $c_{\xi-1}$ that satisfies

$$\eta = -\frac{c_{\xi-1} f(\xi - 1)}{\sum_{j=\xi}^{N} c_j f(j)} (\mathbb{E}[\theta|\xi - 1] + \eta_S),$$

if we cannot find $c_{\xi-1}$ in a way that $c_{\xi-1} \leq 1$, let $c_{\xi-1} = 1$. Next, find $c_{\xi-2}$ that satisfies

$$\eta = -\frac{c_{\xi-2} f\left(\xi-2\right)}{\sum_{j=\xi-1}^{N} c_j f\left(j\right)} (\mathbb{E}[\theta|\xi-2] + \eta_S).$$

If we cannot $c_{\xi-2}$ in a way that $c_{\xi-2} \leq 1$, let $c_{\xi-2} = 1$ and rewrite $c_{\xi-1} = 1$, and continue this process until we get c_1. If we get $c_j > 1$ at some period, rewrite $c_j = c_{j+1} = .. = c_N$ and continue. Let the greatest k such that $c_k < 1$ be γ.

· Step 2. Check if $V = \sum_{j=1}^{N} c_j f\left(j\right) \left(\mathbb{E}[\theta|j] - \eta\right) - \sum_{j=1}^{N} (1-c_j) f\left(j\right) \eta_S \geq \mathbb{E}[\theta]$ and also $V \geq 0$. If both hold, it is done. If it is not let $c_1 = c_2 = .. = c_N = 0$.

The above argument is summarized in the following theorem, which demonstrates that the best equilibrium is found by the procedure when $\Gamma\left(j\right)$ is decreasing.

Theorem 5 *Let $(c_0, c_1, .., c_N)$ be a vector derived from the above procedure. If function $\Gamma\left(j\right)$ is decreasing, there is a unique (in the class of outcome equivalent) DM's utility maximizing equilibrium that is characterized as follows:*

1. $\alpha\left(G, j, \varnothing\right) = c_j,\ \alpha\left(S, j, \varnothing\right) = 1 - c_j$.

2. $\alpha\left(G, j, G^t\right) = 1$ if $t \leq j$ and $t \geq 1$.

3. If $m^t \neq G^t,\ \beta\left(R, m^t\right) = 1$.

4. If $t \leq \gamma,\ \beta\left(A, G^t\right) = \delta/V$. If $t \geq \gamma + 1,\ \beta\left(A, G^t\right) = 1$.

In this equilibrium, each type of sender mixes at period one whether to communicate a good piece of evidence or to give up persuasion by being silent. Once he chooses to communicate a piece of good evidence, he does so until he runs out of it. At each period, say t, the DM can screen out exactly type t sender by continue. The way that type t sender mixes at period one makes the DM's expected benefit from screening out type t and cost of communication equal with each other at period t.[25]

5 Comparative Statics

In this section, we examine the effect of changes in the model's cost parameters (η, η_S, δ) on the equilibrium. In order to do this, hereafter we focus solely on the best equilibrium for the decision maker, and hence from the analysis of section 6, we focus on the best benchmark strategy equilibrium, where value functions are denoted with superscript "*". We have the next theorem, whose proof is easy and thus omitted:

Theorem 6 *1. Fix (η_S, δ). Suppose that the prior of the proposal is bad, i.e., $\mathbb{E}[\theta] < 0$. Then $\mathbb{E}_j[V_S^*\left(\varnothing, j\right)]$ is a step function of η and there is a threshold value of η under which it is increasing, and above which it is zero.*

2. Fix (η_S, δ). Suppose that the prior of the proposal is good, i.e., $\mathbb{E}[\theta] > 0$. Then $\mathbb{E}_j[V_S^*\left(\varnothing, j\right)]$ is a step function of η and there is a threshold value of η under which it is increasing, and above which it is V.

[25] The equilibrium characterized in the theorem has a dropping vector such that $d_j^1 = 1 - c_j,\ d_j^k = 0$ for all $k \in \{2, .., j\}$, and $d_j^{j+1} = 1$.

If the prior of the proposal is bad, the DM whose communication cost η is very high, does not talk with the sender and just reject the proposal. For the sender, this is the worst case because he has no chance of persuading her. The best DM for the sender is the DM whose communication cost is low enough to communicate but not too low to be willing to communicate long time. The DM may communicate as long as $\Gamma(j)$ defined in the previous section exceeds η and thus the maximum length of persuasion is decreasing with η. In this case of the prior of the proposal is bad, the expected payoff of the sender is non-monotonic. Note that the relation between $\mathbb{E}[V_S(\varnothing, j)]$ and η is a step function whose values depends on the length of persuasion.

Figure 2 describes the relation when $\Gamma(j)$ is decreasing, in which the length of persuasion is determined by (14). Because $\Gamma(j)$ is decreasing, as we gradually increase η from zero, the length of persuasion decreases one by one, and thereby increases the expected payoff of the sender.

On the other hand, if the proposal is ex-ante good and the DM has a very high communication cost, the DM does not require evidence from the sender and just rubber-stamps the proposal. For the sender, this is the best possible case in which his expected payoff is maximized. Therefore, in such a case of ex-ante good proposal, the expected payoff of the sender becomes monotonic, which is again a step function.

The effect of change in η on the DM's expected payoff is divided into direct and indirect effect, as we have seen in the example of section 3. We have

$$\begin{aligned} V_{DM}^{*}(\varnothing) &= \sum_{j=1}^{N_G(e)-1} \alpha(G,\varnothing,j) f(j)(\mathbb{E}[\theta|1]-\eta) + \sum_{j=N_G(e)}^{N} f(j)(\mathbb{E}[\theta|1]-\eta) \\ &\quad -\eta_S \sum_{j=1}^{N_G(e)-1} \{1-\alpha(G,\varnothing,j)\} f(j), \end{aligned}$$

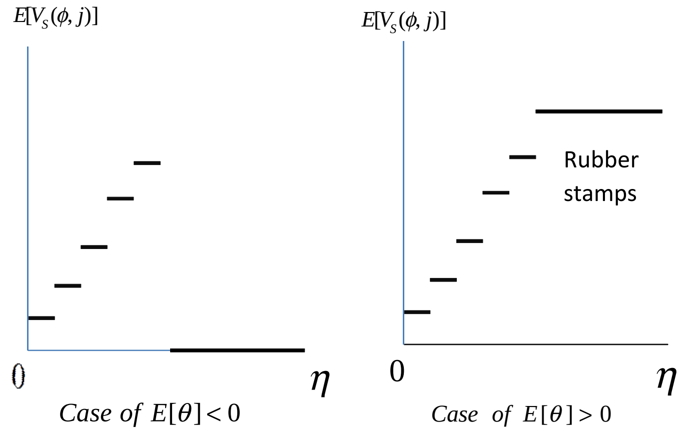
since the sender will communicate a piece of good evidence or being silent at period 1. Hence we have $\frac{\partial V_{DM}^{*}(\varnothing)}{\partial \eta} =$

$$\underbrace{-\sum_{j=1}^{N_G(e^*)-1} \alpha(G,\varnothing,j) f(j) - \sum_{j=N_G(e^*)}^{N} f(j)}_{\text{Direct effect }(-)} + \underbrace{\sum_{j=1}^{N_G(e^*)-1} \frac{\partial \alpha(G,\varnothing,j)}{\partial \eta} f(j)(\mathbb{E}[\theta|1]-\eta+\eta_S)}_{\text{Indirect effect }(-)}.$$

We can see that the direct effect is negative, and are able to show that indirect effect is also negative (see Appendix). Interpretation of the direct effect is straightforward: it just reduces the cost of communication at period one. The indirect effect comes from the later periods. The reduction in η makes it possible to make the DM indifferent between acceptance and continue at each period with a small benefit from screening and hence with a high dropping of bad senders at period 1.

Another important implication is that, assuming that the best equilibrium always holds, the probability that the decision maker makes an wrong decision, either choosing A when the sender type is less than ξ (type II error) or choosing R when the sender type is bigger than ξ (type I error), monotonically converges to zero as η converges to zero. Thus by denoting the probability by $F(\eta, \eta_S)$, the next theorem follows, where its proof is omitted:

Figure 2: the sender's expected payoff



Theorem 7 *Fix δ. Then,*

$$\lim_{\eta \downarrow 0} \sup_{\eta_S \leq \eta} \mathcal{F}\left(\eta, \eta_S\right) \to 0.$$

To see the theorem, first note that the probability that the decision maker makes the wrong decision of accepting the bad proposal, type II error, is smaller than $\sum_{j=1}^{N_G(e^*)-1} \alpha\left(G, \varnothing, j\right) f\left(j\right)$. It is easily seen by (8) that the absolute value of it is decreasing. On the other hand, in the best equilibrium, type I error never happens when $\mathbb{E}[\theta] > 0$. Even when $\mathbb{E}[\theta] \leq 0$, type I error never happens as long as the DM chooses C at period 0, which is the case when η is sufficiently small.

We next consider comparative statics with respect to the sender's communication costs. It is easy to see from the construction of equilibrium that the DM's expected payoff is invariant with the sender' cost of communication. On the other hand, the sender's expected payoff can be naturally shown to be decreasing with his cost of communication.

Theorem 8 *Fix (η, η_S). Then $V_{DM}^*\left(\varnothing\right)$ is constant with respect to δ and $\mathbb{E}[V_S^*\left(\varnothing, j\right)]$ is strictly decreasing with δ.*

The reason for $\mathbb{E}[V_S^*\left(\varnothing, j\right)]$ being strictly decreasing with δ is easy to see. From Claim 3, the low type sender's expected payoff is 0, irrespective of his communication cost δ, which comes from the fact that acceptance probability will adjust in an equilibrium. However, the acceptance probability at period $N_G\left(e^*\right)$, which is 1, cannot adjust with the change in δ, which implies that an increase in δ decreases the high type sender's expected payoff. These also tell that a decrease in δ lengthens the expected time before acceptance.

6 Commitment

In this section, we examine whether the DM can be better off by making a commitment if she can write down a contingent plan to follow. If we think of the DM as an organization which is frequently making decisions based on the advice of concerned parties, this question is particularly important for designing the rule used to handle this advice. We consider two different forms of commitment.

Optimal Stochastic Commitment

The answer to the problem, however, is easy if the DM is allowed to make commitment in a very sophisticated way. In fact, it is easy to see that the following method of commitment, if possible, makes the DM better off for sure: the DM accepts the offer with probability δ/V (or actually, slightly lower than) each time a piece of good evidence is shown, until enough good evidence is shown at which point she accepts the offer for sure. It follows that if the sender does not have enough good pieces of evidence to show, he remains silent from the beginning. Once the DM knows that, she has an incentive to accept as soon as possible. The probability δ/V is the largest probability of acceptance that can make the screening of sender types possible. Actually, it can be shown that this is the optimal commitment that the DM can make.

Theorem 9 *The optimal commitment takes the following form: the DM accepts the proposal with probability δ/V each time the sender communicates a piece of good evidence until κ pieces of good evidence are communicated, where κ is the number characterized by*

$$\begin{aligned}\kappa &= \arg\max_k \sum_{j\geq k} f(j)\,\mathbb{E}[\theta|j] \\ &\quad - \sum_{j\geq k} f(j)\,[\sum_{n\geq 1}^{k} n\eta\left(1-\frac{\delta}{V}\right)^{n-1}\frac{\delta}{V} + k\eta\left(1-\frac{\delta}{V}\right)^{\kappa-1}] - \eta_S \sum_{j<k} f(j)\,.\end{aligned}$$

An important point is that the stochastic commitment is actually a Pareto improvement from the best equilibrium. This follows because in the best equilibrium, the low type sender's expected payoff is zero, and it can be shown that the length of communication is shorter in the commitment case than in the best equilibrium, which makes the sender better-off. This tells us that persuasion game involves an inevitable waste of time or energy, which can be mitigated by the commitment.

Optimal Limited (Non-Stochastic) Commitment

Although stochastic commitment attains a desirable outcome compared to the best equilibrium, making stochastic commitment may be difficult because the agent cannot verify the DM's behavior and the DM cannot prove that she is actually following the committed plan. In order to consider such a case, in particular, we think of the following form of commitment that is easier to make: she decides to listen to the sender for predetermined length of time, say τ, as long as good pieces of evidence are shown. If she is shown τ pieces of good evidence in a row, she accepts the offer, while she rejects the offer as soon as she is shown other evidence or silence. If the DM makes such a commitment, it is optimal for the sender types lower than τ to remain silent at period 1 and get rejected, because they know that they cannot persuade the DM. We call this type of commitment "limited commitment" hereafter.

The optimization problem the DM has to solve when she makes the limited commitment is as follows:

$$\begin{aligned}\max_k \Upsilon_{DM}(k) &= \max_k \{\sum_{j\geq\kappa} f(j)\,(\mathbb{E}[\theta|j]-\kappa\eta) - \eta_S \sum_{j<\kappa} f(j)\}, \\ \text{subject to } \kappa\delta &\geq V.\end{aligned}$$

With probability $\sum_{j\geq\kappa} f(j)$, the sender is high type and the DM has to pay the communication cost of $\kappa\eta$. Otherwise, the sender is low type in which case she will pay just a period cost of silence. We have a participation constraint for the sender, $\kappa\xi \geq V$. Unless this condition is satisfied, the sender does not try to persuade the DM by paying the communication cost.

We have the following result, which shows that the DM is better off by making a limited commitment if the sender's persuasion gain V is high enough relative to his communication cost.

Theorem 10 *Suppose that $V \geq \delta N_G(e^*)$,*[26] *where $N_G(e^*)$ is the length of persuasion of the best equilibrium. If $\beta(C, \varnothing) = 1$ in the best equilibrium, the DM prefers to make limited commitment, i.e.,* $\max_k \Upsilon_{DM}(k) > V^*_{DM}(\varnothing)$.

This result follows from the same reason as we discussed in Section 4. In the best equilibrium, an optimal strategy of the decision maker, given the sender's strategy, is to require further pieces of evidence until enough good evidence is communicated, and otherwise reject. This prevents bad senders from communicating good evidence and then giving up, which causes the DM to incur communication costs.

Contrary to the stochastic commitment, the limited commitment is not a Pareto improvement from the best equilibrium. This follows because in the best equilibrium, the high type sender has a chance of succeeding at persuasion quickly, while he has to communicate for a certain amount of time in a limited commitment case. Because the low type sender's expected payoff is zero both in the best equilibrium and with limited commitment, it can happen that the sender is worse-off in a limited commitment case than in the best equilibrium.[27]

The above result, however, can only be guaranteed if $V \geq \delta N_G(e^*)$, i.e., the sender is willing to pay the persuasion cost in order to induce his preferred action from the DM even if it takes $N_G(e^*)$ periods to communicate for sure. Once this condition is violated, it is possible to have a situation where the DM prefers to play the persuasion game instead of making a limited commitment. An example is shown in the following claim.

Claim 4 *Suppose that the parameter values of the model are as follows: $N = \xi = 2$. $2\delta > V$, $\mathbb{E}[\theta|j \geq 1] > 0$, and the DM's communication cost η is small to the extent that (5) is satisfied. Then, the DM prefers to play the best equilibrium than the limited commitment, i.e.,* $V^*_{DM}(\varnothing) > \max_k \Upsilon_{DM}(k)$.

The proof is easy. Actually, in the setting above, the best limited commitment is to require only one good piece of evidence. If she requires two pieces, no sender type tries to persuade her. Hence she can only require at most one piece of good evidence in the limited commitment, which gives her the same expected payoff as playing the game with the equilibrium of $N_G(e) = 1$. Then the claim follows from Theorem 4, which states that the equilibrium that attains the highest expected payoff for the DM has the longest length of persuasion.

[26] Note that $N_G(e^*)$ is an endogenous variable. Another sufficient condition that uses only exogenous variable is $V \geq \delta\xi$, which is stronger because $\xi \geq N_G(e^*)$.

[27] It can be shown that the length of persuasion is shorter in the commitment case than in the best equilibrium.

More generally, in the best equilibrium, we may have $\delta N_G(e^*) > V$, which means that the sender communicates for too much length and pays more persuasion cost than what he can get (V) if the decision maker postpones the decision the most. This makes the DM possible to extract more information from the sender, relative to the case of limited commitment where the sender is perfectly knowledgeable about the outcome of the persuasion and hence never pay the communication cost excessively.

7 Conclusion

In this study, we created a model that describes the dynamic process of persuasion. We show that the equilibrium necessary involves probabilistic behavior from both parties, and we characterized the set of Pareto efficient equilibria and the best equilibrium for the decision maker.

Although we provided entrepreneur-venture capitalist relation as a primary example, there are a lot of real world examples that fit out model. Glazer and Rubinstein (2004) provide a number of nice examples of persuasion through hard evidence.[28] Those include, for example, the case in which a worker wishes to be hired by an employer for a certain position. The worker tells the employer about his previous experience and the employer wishes to hire the worker if his ability is above a certain level.

It may be interesting to extend the model in a way that parameters η and δ, which represent players' costs of communication, have nondegenerate distributions and also are private information. Then, we will obtain more complicated strategic interactions because the fact that game did not terminate until a particular period conveys some information about the players' types. This gives our game an additional flavor of Fudenberg and Tirole's (1986) war of attrition model.

Obviously, this is just a first step for a deeper understanding of the process of persuasion. There are a lot of questions that cannot be addressed by this study. These include interesting questions such as 1. In which order should pieces of evidence be released when each piece of evidence has a different value? 2. If the sender is allowed to show multiple pieces of evidence at a time, how does this change the nature of persuasion? 3. Can we render a reasonable explanation for why sometime a persuader reveals unfavorable information? Those questions are left up to future research.

8 APPENDIX

In the following, we use the following notations.

T : The set of terminal message history that can be reached with strictly positive probability, that is, $T = \{m^t \in \Delta \mid \beta(C, m^t) = 0\}$.

$P(m^t)$: The set of types of sender that follows message history m^t with strictly positive probability, that is, if $j \in P(m^t)$ then $\Pi_{s=1}^t \alpha(m_s, m^{s-1}, j) > 0$.

$\succ$: Incomplete order on M defined as $m^s \succ m^t$ if and only if $m^s = (m^t, m^s_{t+1})$ for some m^s_{t+1} such that $s \geq t+1$, i.e., m^s is a continuation from m^t.

[28] They work on a setting that the DM is restricted to check only one piece of evidence. In this sense, they think of the case that players face a very tight constraint in communication relative to our model.

$\alpha(m^\tau, j)$: The probability that type j sender follows communication history m^τ, that is, $\alpha(m^\tau, j) = \Pi_{s=1}^{\tau}\alpha(m_s, m^{s-1}, j)$.

$\alpha\left(m_{t-1}^{\tau}, j\right)$: The probability that type j sender follows message history m_{t-1}^{τ} from period $t-1$, that is, $\alpha\left(m_{t-1}^{\tau}, j\right) = \Pi_{s=t}^{\tau}\alpha(m_s, m^{s-1}, j)$.

8.1 Proof for Section 2

Proof for Lemma 1 and 2: We first prove the uniqueness of V_{DM}. Let (φ, B) be given. Suppose that we have two value functions V_{DM} and V'_{DM} that satisfy conditions (1) and (2). Let

$$W = \{m^t | V_{DM}(m^t) \neq V'_{DM}(m^t)\},$$

which is the set of message history such that two value functions take different values. To get a contradiction, suppose that $W \neq \varnothing$. Then there must be some $m^\tau \in W$ such that

$$\begin{aligned} V_{DM}(m^\tau) &> \max_{a\in\{A,R\}} \mathbb{E}[U_{DM}(a, j, m^\tau)|m^\tau] \\ \text{and } V'_{DM}(m^\tau) &= \max_{a\in\{A,R\}} \mathbb{E}[U_{DM}(a, j, m^\tau)|m^\tau], \end{aligned} \tag{15}$$

or

$$\begin{aligned} V_{DM}(m^\tau) &= \max_{a\in\{A,R\}} \mathbb{E}[U_{DM}(a, j, m^\tau)|m^\tau] \\ \text{and } V'_{DM}(m^\tau) &> \max_{a\in\{A,R\}} \mathbb{E}[U_{DM}(a, j, m^\tau)|m^\tau], \end{aligned} \tag{16}$$

where we denote $\sum_{n=0}^{N} B_n(m^t) U_{DM}(a, j, m^t)$ by $\mathbb{E}[U_{DM}(a, j, m^t)|m^t]$.

To see this, note that if neither holds, $V_{DM}(m^\tau) > \max_{a\in\{A,R\}} \mathbb{E}[U_{DM}(a, j, m^\tau)|m^\tau]$ and $V'_{DM}(m^\tau) > \max_{a\in\{A,R\}} \mathbb{E}[U_{DM}(a, j, m^\tau)|m^\tau]$ for all $m^\tau \in W$. Those, respectively, imply that

$$\begin{aligned} V_{DM}(m^\tau) &= \mathbb{E}[V_{DM}(m^\tau, m_{\tau+1})|m^\tau)] = \sum_{m_{t+1}} \varphi(m_{\tau+1}|m^\tau) V_{DM}(m_\tau, m^\tau) \\ \text{and } V'_{DM}(m^\tau) &= \mathbb{E}[V'_{DM}(m^\tau, m_{\tau+1})|m^\tau)] = \sum_{m_{t+1}} \varphi(m_{\tau+1}|m^\tau) V'_{DM}(m_\tau, m^\tau). \end{aligned}$$

Since $m^\tau \in W$, we let $V_{DM}(m^\tau) > V'_{DM}(m^\tau)$, without loss of generality. Then above relations imply that there is some $m_{\tau+1}$ such that

$$V_{DM}(m^\tau, m_{\tau+1}) > V'_{DM}(m^\tau, m_{\tau+1}) > \max_{a\in\{A,R\}} \mathbb{E}[U_{DM}(a, j, (m^\tau, m_{\tau+1})|m^\tau, m_{\tau+1}].$$

By continuing the same argument, we have a sequence $\{m_s\}_{s=t}^{\infty}$ such that

$$V_{DM}(m^s) \leq V_{DM}(m^s, m_{s+1}) \text{ for all } s \geq \tau.$$

From (2), we have to have $\lim_{s\to\infty} V'_{DM}(m^s, m_{s+1}) = \lim_{s\to\infty} -\eta\{N_G(m^s) + N_B(m^s)\} = -\infty$, but this contradicts $V_{DM}(m^\tau) > \max_{a\in\{A,R\}} \mathbb{E}[U_{DM}(a, j, m^\tau)|m^\tau]$.

Without loss of generality, let (15) holds. For every history m^t that can be reached from m^τ with strictly positive probability, we can find the smallest $s \leq t$ such that $V_{DM}(m^s) =$

$\max_{a\in\{A,R\}}\mathbb{E}[U_{DM}(a,j,m^s)|m^s]$. To see this, note that if it is not, we have a sequence $\{m_s\}_{s=\tau}^{\infty}$ such that $V_{DM}(m^s) \leq V_{DM}(m^s, m_{s+1})$ for all $s \geq \tau$ and thus $V_{DM}(m^s) \leq \lim_{s\to\infty} V_{DM}(m^s) = -\infty$, which contradicts $V_{DM}(m^\tau) > \max_{a\in\{A,R\}}\mathbb{E}[U_{DM}(a,j,m^\tau)|m^\tau]$. Let the set of such history Λ, and probability measure on Λ generated by φ be ω. Then we have $V_{DM}(m^\tau) = \int V_{DM}(s)\,d\omega(s)$. For each $s\in\Lambda$, we must have $V'_{DM}(m^s) \geq \max_{a\in\{A,R\}}\mathbb{E}[U_{DM}(a,j,m^s)|m^s] = V_{DM}(m^s)$, and hence we have $\int V'_{DM}(s)\,d\omega(s) \geq \int V_{DM}(s)\,d\omega(s)$. However, it must hold that $V'_{DM}(m^\tau) \geq \int V'_{DM}(s)\,d\omega(s)$ from the definition of the value function, we have $V'_{DM}(m^\tau) \geq V_{DM}(m^\tau)$, which is a contradiction. This shows the uniqueness of the value function V_{DM}.

Next, we prove the uniqueness of V_S. To get a contradiction, suppose that we have two different value functions, and let $V_S(m^t, j) > V'_S(m^t, j)$ for some j and m^t. Make the sequence $\{m^s\}_{s=t}^{\infty}$ by

$$\begin{aligned} V_S\left(m^t, x\right) &= \beta\left(A, m^t\right)\left(V - C_S\left(m^t\right)\right) \\ &\quad -\beta\left(R, m^t\right) C_S\left(m^t\right) + \beta\left(C, m^t\right) V_S\left(\left(m^{t+1}, a\right), j\right). \end{aligned}$$

Then we have

$$\begin{aligned} &\lim \Pi_{t=0}^{\infty}\beta\left(C, m^t\right) V_S\left(m^{t+1}\right) - \lim \Pi_{t=0}^{\infty}\beta\left(C, m^t\right) V'_S\left(m^{t+1}, j\right) \\ > \;& V_S\left(m^t, j\right) - V'_S\left(m^t, j\right) > 0. \end{aligned}$$

, which contradicts (2).

Next, we will prove the existence of V_{DM}. To shorten the notation, denote by $g(m^t)$ the highest value of expected utility of the DM when she decides whether to accept or reject, i.e., $g(m^t) = \max_{a\in\{A,Q\}}\mathbb{E}[U(a,j,m^t)|m^t]$. Take a particular m^t and fix it. Make the sequence of real numbers $V_0(m^t), V_1(m^t), \ldots$ as follows. Let

$$V_0\left(m^t\right) = g\left(m^t\right),\;\; V_1\left(m^t\right) = \max\{g\left(m^t\right), \sum_{m^{t+1}} \varphi\left(m^{t+1}|m^t\right) g\left(m^{t+1}\right),$$

and $V_2(m^t) =$

$$\max\{g\left(m^t\right), \sum_{m_{t+1}} \max\{\varphi\left(m^{t+1}|m^t\right) g\left(m^{t+1}\right), \sum_{m_{t+2}} \{\varphi\left(m^{t+2}|m^{t+1}\right) g\left(m^{t+2}\right)\},$$

and so on. That is, $V_k(m^t)$ is constructed by $V_{k-1}(m^t)$ by replacing terms

$$\varphi\left(m^{t+k-1}|m^{t+k-2}\right) g\left(m^{t+k-1}\right)$$

with

$$\max\{\varphi\left(m^{t+k-1}|m^{t+k-2}\right) g\left(m^{t+k-1}\right), \sum_{m_{t+k-1}} \{\varphi\left(m^{k+t}|m^{k+t-1}\right) g\left(m^{k+t}\right)\}.$$

Obviously, the sequence $V_n(m^t)$ is an increasing sequence with each satisfies $V_n(m^t) \leq 1 - \eta\{N_G(m^t) + N_B(m^t)\}$. Hence it converges to some value $V_\infty(m^t) \leq 1 - \eta\{N_G(m^t) + N_B(m^t)\}$. Let this value be $V_{DM}(m^t)$, and do this for all elements in H. It is a routine work to verify that those satisfy the condition for being the value function.

To prove the existence of $V_S(m^t,j)$, pick a pair (m^t,j) and fix it. Again, define the sequence $V_0(m^t,j)$, $V_1(m^t,j)$, .. as follows:

$$V_0\left(m^t,j\right)=\beta\left(A,m^t\right)\left\{V-C_S\left(m^t\right)\right\}-\beta\left(Q,m^t\right)C_S\left(m^t\right)-\beta\left(C,m^t\right)C_S\left(m^t\right),$$

$$\begin{aligned}V_1\left(m^t,j\right) &= \beta\left(A,m^t\right)\left(V-C_S\left(m^t\right)\right)-\beta\left(R,m^t\right)C_S\left(m^t\right)\\ &\quad -\beta\left(C,m^t\right)[\max_{a\in M(m^t,j)}\{\beta\left(A,\left(m^t,a\right)\right)\left(V-C_S\left(m^t,a\right)\right)\\ &\quad -\beta\left(R,\left(m^t,a\right)\right)C_S\left(m^t,a\right)-\beta\left(C,m^t\right)C_S\left(m^t,a\right)\}],\end{aligned}$$

and so on. That is, $V_{k+1}(m^t,j)$ is constructed by $V_k(m^t)$ by replacing terms $\beta\left(C,m^{t+k}\right)C\left(m^{t+k},a\right)$ with

$$\begin{aligned}&\max_{a\in M\left(m^{t+k},j\right)}[\beta\left(A,\left(m^{t+k},a\right)\right)\left(V-C_S\left(m^{t+k},a\right)\right)-\beta\left(A,\left(m^{t+k},a\right)\right)\left(V-C_S\left(m^{t+k},a\right)\right)]\\ &-\beta\left(C,\left(m^{t+k},a\right)\right)C_S\left(m^{t+k},a\right)\}.\end{aligned}$$

Then obviously, the sequence $V_n(m^t,j)$ is an increasing sequence with each satisfies

$$-C_S\left(m^t\right)\le V_n\left(m^t,j\right)\le V-C_S\left(m^t\right)\le V.$$

Hence it converges to some value $V_\infty(m^t)$. Let this value be $V_S(m^t,j)$, and do this for all elements in M. It is a routine work to verify that those satisfy the condition for being the value function. *Q.E.D.*

Proof of Claim 2: Take an arbitrary PBE and let it $\left(\widehat{\alpha},\widehat{\beta},\widehat{\varphi},\widehat{B}\right)$. We will construct a sequence of totally mixed strategy $(\alpha^\lambda,\beta^\lambda,\varphi^\lambda,B^\lambda)$, with $\lambda\in\boldsymbol{N}_+$ (and $\lambda\ge 2$) that converges to $\left(\widehat{\alpha},\widehat{\beta},\widehat{\varphi},\widehat{B}\right)$ as $\lambda\to\infty$. More precisely, we show that there is a sequence $\left(\alpha^\lambda,\beta^\lambda,\varphi^\lambda,B^\lambda\right)$ such that for each $m^t\in M$ and j, it converges to $\left(\widehat{\alpha}(m^t,j),\widehat{\beta}(m^t),\widehat{\varphi}(m^t),\widehat{B}(m^t)\right)\in R^{9+N}$ in a way such that $\beta^\lambda(m^t)\geqslant 0$ as a vector in R^3, $\alpha^\lambda(S,m^t,j)>0$, $\alpha^\lambda(G,m^t,j)>0$ if $j>N_G(m^t)$, $\alpha^\lambda(B,m^t,j)>0$ if $N-j>N_B(m^t)$, and $\varphi^\lambda(m^t)$ and $B^\lambda(m^t)$ are induced by Bayes rule.

In order to do this, we first specify the sequence of sender's first period strategy as follows. We will choose, for each $m_1\in M$, sufficiently large number $\Psi(m_1)$, and a function $\varepsilon^\lambda(\cdot,\cdot,\varnothing)$: $M\times N\to[0,1]$. Let the following conditions are satisfied: first, if $j\notin M(m_1,j)$, $\varepsilon^\lambda(m_1,\varnothing,j)=0$. Next think of m_1 such that $\sum_{n=0}^{N}\alpha(m_1,\varnothing,j)=0$ (hence m_1 is an off-equilibrium message). If $N_G(m_1)<\xi$, let it satisfies the followings: for j such that $m_1\in M(\varnothing,j)$ and $j<\xi$, we have

$$\frac{f(j)\,\varepsilon^\lambda(m_1,\varnothing,j)}{\sum f(n)\,\varepsilon^\lambda(m_1,\varnothing,n)}=\frac{1}{|\{n|m_1\in M(m_1,n)\text{ and }n<\xi\}|}\left(\frac{\lambda-1}{\lambda}\right),$$

and for j such that $m_1\in M(\varnothing,j)$ and $j\ge\xi$,

$$\frac{f(j)\,\varepsilon^\lambda(m_1,\varnothing,j)}{\sum\limits_{m_1\in M(m_1,n)}f(n)\,\varepsilon^\lambda(m_1,\varnothing,n)}=\frac{1}{|\{n|m_1\in M(m_1,n)\text{ and }n\ge\xi\}|}\frac{1}{\lambda}.$$

If $N_G(m_1) \geq \xi$ (and thus $\xi = 1$ and $m_1 = G$), let it satisfies

$$\varepsilon^\lambda(m_1, \varnothing, j) = 1 \text{ for } j \geq 1 \text{ and } \varepsilon^\lambda(m_1, \varnothing, x) = 0 \text{ for } j < 0.$$

On the other hand, for each m_1 such that $\sum_{j=0}^{N} \alpha(m_1, \varnothing, j) > 0$ (hence m_1 is an on-equilibrium message), let it satisfies

$$\varepsilon^\lambda(m_1, \varnothing, j) = \frac{1}{\Psi(m_1)\lambda} \text{ if } \alpha(m_1, \varnothing, j) = 0,\ \varepsilon^\lambda(m_1, \varnothing, j) = 0 \text{ if } \alpha(m_1, \varnothing, j) > 0.$$

Finally, ε^λ takes only sufficiently small numbers. More precisely, let it satisfies

$$\sum_{m_1 \in M(x,\varnothing)} \varepsilon^\lambda(m_1, \varnothing, x) < \min_{a \in M(x,\varnothing),\ \alpha(m_1,\varnothing,x)>0} \alpha(m_1, \varnothing, x).$$

By using $\varepsilon^\lambda(m_1, \varnothing, j)$, we will construct the sender's first period's strategy $\alpha^\lambda(m_1, \varnothing, j)$ as follows. For m_1 such that $\alpha(m_1, \varnothing, j) = 0$, let $\alpha^\lambda(m_1, \varnothing, j) = \varepsilon^\lambda(m_1, \varnothing, j)$ (hence $\alpha(m_1, \varnothing, j) = 0$ if $m_1 \notin M(j, \varnothing)$) and for m_1 such that $\alpha(m_1, \varnothing, j) > 0$, let

$$\alpha^\lambda(m_1, \varnothing, j) = \alpha(m_1, \varnothing, j) - \frac{1}{|\{m_1 | \alpha(m_1, \varnothing, j) > 0\}|} \sum_{m_1 \in M} \varepsilon^\lambda(m_1, \varnothing, j). \tag{17}$$

Note that $\sum_{m_1 \in M(j,\varnothing)} \widehat{\alpha}^\lambda(m_1, \varnothing, j) = 1$, and it constitutes the totally mixed first period strategy for the sender.

In order to construct a sequence of the first period's strategy for the DM, let $\varkappa(m^t) = |\{a \in \{A, R, C\} \,|\beta(a, m^t) > 0\}|$, which is the number of actions that DM takes with strictly positive probability after message history m^t (and hence less than three). Then for $m_t \in \Delta$, let it be

$$\begin{aligned} \beta^\lambda(a, m_1) &= \widehat{\beta}(a, m_1) - \frac{1}{\varkappa(m_1)\lambda} \text{ if } \widehat{\beta}(a, m_1) > 0, \\ \text{and } \beta^\lambda(a, m_1) &= \frac{1}{\varkappa(m_1)\lambda} \text{ if } \widehat{\beta}(a, m_1) = 0. \end{aligned}$$

And for $m_1 \notin \Delta$, let it be

$$\begin{aligned} \beta^\lambda(Q, m_1) &= \frac{2\lambda - 1}{2\lambda} \text{ and } \beta^\lambda(A, m_1) = \beta^\lambda(C, m_1) = \frac{1}{2\lambda} \text{ if } N_G\left(m^t\right) < \xi, \\ \text{and } \beta^\lambda(A, m_1) &= \frac{2\lambda - 1}{2\lambda} \text{ and } \beta^\lambda(Q, m_1) = \beta^\lambda(C, m_1) = \frac{1}{2\lambda} \text{ if } N_G\left(m^t\right) \geq \xi. \end{aligned}$$

$$B_j^\lambda(m_1) = \frac{f(j)\,\alpha^\lambda(m_1, \varnothing, j)}{\sum_{n=0}^{N} f(n)\,\alpha^\lambda(m_1, \varnothing, j)} \textit{ and } \varphi^\lambda(m_1 | \varnothing) = \sum_j f(j)\,\alpha^\lambda(m_1, \varnothing, j).$$

The idea is to make DM's strategy put strictly high probability of rejection (acceptance) after every off-equilibrium messages with low (high) number of good evidence in the original equilibrium.

Next we define the totally mixed strategy for period 2. Fix $m_1 \in H_1$. For each m_2, choose sufficiently large number $\Psi(m_1, m_2)$, and function $\varepsilon^\lambda(\cdot, \cdot, m_1) : M \times N \to [0,1]$. Let the following conditions are satisfied: First, if $j \notin M(m_1, j)$, $\varepsilon^\lambda(m_1, j, \varnothing) = 0$. Next think of the case in which $\sum_{n=0}^{N} \alpha(m_1, n, \varnothing) = 0$. If $N_G(m_1, m_2) < \xi$, let it satisfies the followings: for j such that $m_2 \in M(m_1, j)$ and $j < \xi$, we have

$$\frac{B_j^\lambda(m_1)\varepsilon^\lambda(m_2, m_1, j)}{\sum B_n^\lambda(m_1)\varepsilon^\lambda(m_2, m_1, j)} = \frac{1}{|\{n | m_1 \in M((m_1, m_2), n) \text{ and } n < \xi\}|}\left(\frac{\lambda - 1}{\lambda}\right), \tag{18}$$

and if $j \geq \xi$ and $m_2 \in M(m_1, j)$, let it satisfies

$$\frac{B_j^\lambda(m_1)\varepsilon^\lambda(m_2, m_1, j)}{\sum_{m_1 \in M(m_1, n)} B_j^\lambda(m_1)\varepsilon^\lambda(m_2, m_1, j)} = \frac{1}{|\{n | m_1 \in M((m_1, m_2), n) \text{ and } n \geq \xi\}|}\frac{1}{\lambda}. \tag{19}$$

On the other hand, for each m_2 such that $\sum_{j=0}^{N} \alpha(m_2, m_1, j) > 0$,

$$\varepsilon^\lambda(m_2, m_1, j) = \frac{1}{\Psi(m_1, m_2)\lambda} \text{ if } \alpha(m_2, m_1, j) = 0, \ \varepsilon^\lambda(m_2, m_1, j) = 0 \text{ if } \alpha(m_2, m_1, j) > 0,$$

and moreover,

$$\sum_{a \in M(j, (m_1, m_2))} \varepsilon^\lambda(m_2, m_1, j) < \min_{a \in M(j, (m_1, m_2)), \ \alpha(m_2, j, m_1) > 0} \alpha(m_2, m_1, j).$$

By using $\varepsilon^\lambda(m_2, m_1, j)$, construct $\alpha^\lambda(m_2, m_1, j)$ as follows. For m_2 such that $\alpha(m_2, m_1, j) = 0$, let $\alpha^\lambda(m_2, m_1, j) = \varepsilon^\lambda(m_2, m_1, j)$ and for m_2 such that $\alpha(m_2, m_1, j) > 0$, let

$$\alpha^\lambda(m_2, m_1, j) = \alpha(m_2, m_1, j) - \frac{1}{|\{m_1 | \alpha(m_2, m_1, j) > 0\}|}\sum_j \varepsilon^\lambda(m_2, m_1, j). \tag{20}$$

Note that $\sum_a \widehat{\alpha}^\lambda(m_2, m_1, j) = 1$, and it constitutes the totally mixed first period strategy for the sender. Let

$$B_j^\lambda(m_1, m_2) = \frac{B_j^\lambda(m_1)\alpha^\lambda(m_2, m_1, j)}{\sum_{n=0}^{N} B_j^\lambda(m_1)\alpha^\lambda(m_2, m_1, j)} \text{ and } \varphi(m_2 | m_1) = \frac{\sum_{n=0}^{N} B_n(m_1)\alpha(m_2, m_1, j)}{\sum_{n=0}^{N} B_n(m_1)\alpha(m_2, m_1, j)}.$$

For the second period's strategy for the DM, for $m^2 = (m_1, m_2) \in \Delta$, let it be

$$\begin{aligned} \beta^\lambda(a, m^2) &= \widehat{\beta}(a, m^2) - \frac{1}{\varkappa(m^2)\lambda} \text{ if } \widehat{\beta}(a, m^2) > 0, \\ \text{and } \beta^\lambda(a, m^2) &= \frac{1}{\varkappa(m^2)\lambda} \text{ if } \widehat{\beta}(a, m^2) = 0. \end{aligned}$$

And for $m_1 \notin \Delta$, let it be

$$\begin{aligned}\beta^{\lambda}\left(Q, m^2\right) &= \frac{2\lambda-1}{2\lambda} \text{ and } \beta^{\lambda}\left(A, m^2\right) = \beta^{\lambda}\left(C, m^2\right) = \frac{1}{2\lambda} \text{ if } N_G\left(m^2\right) < \xi, \\ \text{and } \beta^{\lambda}\left(A, m^2\right) &= \frac{2\lambda-1}{2\lambda} \text{ and } \beta^{\lambda}\left(Q, m^2\right) = \beta^{\lambda}\left(C, m^2\right) = \frac{1}{2\lambda} \text{ if } N_G\left(m^2\right) \geq \xi.\end{aligned}$$

Strategies after period 3 are constructed inductively, and we eventually get $\left(\alpha^{\lambda}, \beta^{\lambda}, B^{\lambda}, \varphi^{\lambda}\right)$. Using the almost the same procedure, we can construct $\left(\alpha^{\lambda+1}, \beta^{\lambda+1}, B^{\lambda}, \varphi^{\lambda+1}\right)$. In doing this, replace λ with $\lambda+1$, but use the same $\Psi(m)$ for all $m \in M$. Moreover, choose $\varepsilon^{\lambda+1}(m_t, j, h)$ in such an way that $\max \varepsilon^{\lambda+1}(m_t, j, h) < \frac{1}{2}\varepsilon^{\lambda}(m_t, j, h)$, from which we can get $\lim_{\lambda\to\infty} \varepsilon^{\lambda+1}(m_t, j, m) = 0$ for all $j,\ a$, and $m \in M$. It is possible since the left hand side in (18) and (19) are homogeneous of degree 0 with respect to $\varepsilon^{\lambda}(a, j, h)$. Let

$$(\alpha, \beta, \varphi, B) := \lim_{\lambda\to\infty}(\alpha^{\lambda}, \beta^{\lambda}, \varphi^{\lambda}, B^{\lambda}).$$

It is easy to see that $\left(\widehat{\alpha}, \widehat{\beta}, \widehat{\varphi}, \widehat{B}\right) = (\alpha, \beta, \varphi, B)$ on every $h \in \Delta$, and $\Delta = \Delta'$. To see that $\left(\widehat{\alpha}, \widehat{\beta}, \widehat{\varphi}, \widehat{B}\right)$ is an equilibrium, note that for any $m^t \notin \Delta,\ \widehat{V}_{DM}(m^t) \leq V_{DM}(m^t)$ and hence for all m^t such that $m^t \in \Delta$ and $\widehat{\beta}(C, m^t) = 0,\ \widehat{\beta}(C, m^t) = 0$ is also an optimal. This shows the optimality of DM. It is easy to see the optimality of the sender's strategy. *Q.E.D.*

Proof of Proposition 1: Suppose that $m^{\tau} = (m^{\tau-1}, S) \in \Xi$. Then because being silent incurs no cost but can persuade the DM, it must hold that $\alpha(S, m^{\tau-1}, j) = 1$ for all j, which implies that $\varphi(S|m^{\tau-1}) = 1$, and $V_{DM}(m^{\tau-1}) > \mathbb{E}[V_{DM}(m^{\tau-1}, m_{\tau})|m^{\tau-1}]$. This implies $\beta(A, m^{\tau-1}) = 1$, which contradicts $(m^{\tau-1}, S) \in \Xi$. That $m^{\tau} = (m^{\tau-1}, B) \in \Xi$ is impossible follows from Theorem 1 in the next section.

8.2 Proof for Section 4

Proof of Theorem 1: To prove the theorem, suppose that we have an equilibrium in which $\beta(A, (m^t, G)) = 0$ for some $(m^t, G) \in \Delta$. Obviously, it must hold $\beta(C, (m^t, G)) > 0$. Think about period $t+2$, after message history (m^t, G). Then there must be some message m'_{t+2} such that $\beta\left(A, (m^t, G, m'_{t+2})\right) = 0$ and $\left(m^t, G, m'_{t+2}\right) \in \Delta$, since otherwise, we have $V_{DM}(m^t, G, m_{t+2}) = \sum_{j=0}^{N} B_j(m^t) U_{DM}(A, j, m^t)$ for all $(m^t, G, m_{t+2}) \in \Delta$, which implies

$$V_{DM}\left(m^t, G\right) \geq \sum_{j=0}^{N} B_j\left(m^t, G\right) U_{DM}(A, j, \left(m^t, G\right))$$

$$\begin{aligned}&> \sum_{m_{t+2}\in M} \varphi\left(m_{t+2}|\left(m^t, G\right)\right) \sum_{j=0}^{N} B_j\left(m^t, G, m_{t+2}\right) U_{DM}(A, j, \left(m^t, G, m_{t+2}\right)) \\ &= \sum_{m_{t+2}\in M} \varphi\left(m_{t+2}|\left(m^t, G\right)\right) V_{DM}\left(m^t, G, m_{t+2}\right),\end{aligned}$$

which contradicts $\beta(A, (m^t, G)) = 0$. On the other hand, we must have $\beta\left(R, m^t, G, m'_{t+2}\right) < 1$, since if it is the case senders from $P\left(A, (m^t, G, m'_{t+2})\right)$ should have chosen S at period $t+1$ after

m^t, which contradicts with $\left(m^t, G, m'_{t+2}\right) \in \Delta$. This implies that $\beta\left(C, \left(m^t, G, m'_{t+2}\right)\right) > 0$, and hence

$$V_{DM}\left(m^t, G, m'_{t+2}\right) = \sum_{m_{t+3} \in M} \varphi\left(m_{t+3} | m^t, G, m'_{t+2}\right) V_{DM}\left(m^t, G, m'_{t+2}, m_{t+3}\right).$$

Those imply that at period $t+2$, after all the on-equilibrium message, either A is an optimal or C is an optimal, where in the latter case, the probability of acceptance is zero. Take such message and let it m_{t+2}, and denote (m^t, G, m_{t+2}) by m^{t+2}. At period $t+3$ after m^{t+2}, there is no $\left(m^{t+2}, m'_{t+3}\right) \in \Delta$ such that $\beta\left(R, m^t, \left(m^{t+2}, m'_{t+3}\right)\right) = 1$ and $\left(m^{t+2}, m'_{t+3}\right) \in \Delta$, since otherwise, senders from $P\left(m^{t+2}, m'_{t+3}\right)$ should have chosen S at period $t+1$ after m^t, which contradicts $\left(m^{t+2}, m'_{t+3}\right) \in \Delta$. It in turn, implies that $\beta\left(A, m^t, \left(m^{t+2}, m'_{t+3}\right)\right) > 0$ or $\beta\left(C, m^t, \left(m^{t+2}, m'_{t+3}\right)\right) > 0$. In sum we have

$$V_{DM}\left(m^{t+2}, m'_{t+3}\right) = \sum_{j=0}^{N} B_j\left(m^{t+2}, m'_{t+3}\right) U_{DM}(A, j, \left(m^{t+2}, m'_{t+3}\right))$$

or

$$V_{DM}\left(m^{t+2}, m'_{t+3}\right) = \sum_{m_{t+4} \in M} \varphi\left(m_{t+4} | m^{t+2}, m'_{t+3}\right) V_{DM}(m^{t+2}, m'_{t+3}, m_{t+4}).$$

Repeat the same reasoning, we could see that every on-equilibrium history m^τ that is a continuation from (m^t, G), we must have

$$V_{DM}(m^\tau) = \max\{\sum_{j=0}^{N} B_j(m^\tau) U_{DM}(A, j, m^t), \sum_{m_{\tau+1} \in M} \varphi(m_{\tau+1} | m^\tau) V_{DM}(m^\tau, m_{\tau+1})\}.$$

Then, it is easy to see that

$$V_{DM}\left(m^t, G\right) > \sum_{m_{t+2} \in M} \varphi\left(m_{t+2} | m^t, G\right) V_{DM}(m^t, G, m_{t+1}),$$

which contradicts with $\beta(C, (m^t, G)) > 0$. We can apply the same proof to show that $(m^t, B) \in \Delta$ implies $\beta(A, (m^t, B)) > 0$.

To see that $(m^t, G) \in \Delta$ implies $\beta(A, (m^t, G)) \geq \delta/V$, suppose that there is some $(m^t, G) \in \Delta$ such that $\beta(A, (m^t, G)) < \delta/V$. If $\beta(C, (m^t, G)) = 0$, then the sender should S after m^t rather than G, which contradicts to $(m^t, G) \in \Delta$. Hence we have $\beta(C, (m^t, G)) > 0$. Using the same type of argument above, we can see that there must be some type of sender in $P(m^t, G)$ such that he is never accepted after period $t+2$ (he sends S after (m^t, G)). For such type of sender, it is strictly better to send S after m^t, which gives him at least $-\delta\{N_G(m^t) + N_B(m^t)\}$, rather than sending G, which gives him only $\beta(A, (m^t, G)) \cdot V - \delta\{N_G(m^t) + N_B(m^t)\} - \delta < -\delta\{N_G(m^t) + N_B(m^t)\}$.

Next, we see that $(m^t, B) \in \Delta$ implies $\beta(A, (m^t, B)) = \delta/V$. We can apply the same proof above to show that $(m^t, B) \in \Delta$ implies $\beta(A, (m^t, B)) \geq \delta/V$. Hence suppose that $(m^t, B) \in \Delta$ and $\beta(A, (m^t, B)) > \delta/V$. Those and $m^t \in \Delta$ imply that $(m^t, S) \in \Delta$ and $\beta(A, (m^t, S)) = 0$, and thus there must be some type of sender in $P(m^t, S)$ that follows a message history such that he is never accepted after period $t+1$. However rather than that, he can send B after m^t and get the strictly higher expected payoff of

$$\beta\left(A, \left(m^t, B\right)\right) \cdot V - \delta\{N_G\left(m^t\right) + N_B\left(m^t\right)\} - \delta > -\delta\{N_G\left(m^t\right) + N_B\left(m^t\right)\},$$

unless he has no more bad evidence. If he has no more pieces of bad evidence, then it contradicts the optimality of the DM's behavior of not accepting him. Thus, we must have $\beta\left(A,\left(m^t,B\right)\right)\leq\delta/V$. *Q.E.D.*

Proof of Theorem 2: To get a contradiction, suppose that there is some $(m^t,S)\in\Delta$ such that $\beta\left(A,\left(m^t,m\right)\right)>0$. If $(m^t,G)\in\Delta$ (or $(m^t,B)\in\Delta$), from Theorem 1 it must hold that $\beta\left(A,\left(m^t,m\right)\right)>0$ (or $\beta\left(A,\left(m^t,m\right)\right)>0$). In such case, choosing A at period $t+2$ becomes an optimal after all on-equilibrium message at $t+2$, and hence contradicts $\beta\left(C,m^t\right)>0$. Thus we must have $(m^t,G)\notin\Delta$ and $(m^t,B)\notin\Delta$. However if it is the case, $\varphi\left(S|m^t\right)=1$ and the decision maker does not expect belief updating at period $t+2$, which implies $\beta\left(C,\left(m^t,m\right)\right)=1$ and hence $\beta\left(A,\left(m^t,m\right)\right)=0$. *Q.E.D.*

8.3 Proof for Section 5

Proof of Proposition 4: Only if direction: Suppose that we have benchmark strategy equilibrium e with the sender's strategy α, and let $\kappa=N_G\left(e\right)$. Obviously, $\beta\left(A,G^{\kappa+1}\right)=1$ and $\beta\left(A,G^t\right)=\eta/V$ for $t\leq\kappa$. Because for $t\leq\kappa$, A as well as C are optimal for the DM, from $D1$, it must hold that

$$\begin{aligned}\sum_{j=0}^{N}B_j\left(G^t\right)U_{DM}(A,j,G^t) &= \sum_{m\in M}\varphi(m|m^t)V_{DM}\left(m_t,m\right) \\ &= \varphi(G|G^t)V_{DM}\left(G^{t+1}\right)+\varphi(S|G^t)V_{DM}\left(G^t,m\right).\end{aligned} \tag{21}$$

In the benchmark strategy equilibrium, for all $(m,S)\in\Delta$, $V_{DM}\left(G^t,m\right)=-C_{DM}\left(m^t,m\right)=-\eta t-\eta_S$, because $\beta\left(R,\left(G^t,m\right)\right)=1$. Moreover, we have

$$B_j\left(G^\tau\right) = \frac{\Pi_{s=1}^{\tau}\left(1-d_j^s\right)f\left(j\right)}{\sum_{j\geq\tau}\Pi_{s=1}^{\tau}\left(1-d_j^s\right)f\left(j\right)} \tag{22}$$

$$\text{and } \varphi(G|G^\tau) = \frac{\sum_{j\geq\tau+1}\left(1-d_j^{\tau+1}\right)\Pi_{s=1}^{\tau}\left(1-d_j^s\right)f\left(j\right)}{\sum_{j\geq\tau}\Pi_{s=1}^{\tau}\left(1-d_j^s\right)f\left(j\right)}. \tag{23}$$

Then by substituting those into (21), we can see that (8) must hold. Since the optimal action for the DM after $m^1=G$ is A, (9) must hold as well.

If direction: let $\beta\left(A,G^{\kappa+1}\right)=1$ and $\beta\left(A,G^t\right)=\eta/V$ for $t\leq\kappa$, $\alpha\left(B,m^t,j\right)=0$ for all $m^t\in H$ and B and φ satisfy (22), as well as

$$\begin{aligned}B_N\left(m^t\right) &= 1 \text{ for all } m^t \text{ such that } m^t\neq G^{\kappa+1} \text{ and } N_G\left(m^t\right)\geq\xi, \\ B_{N_G(m^t)}\left(m^t\right) &= 1 \text{ for all } m^t \text{ such that } m^t\neq G^{\kappa+1} \text{ and } N_G\left(m^t\right)<\xi, \\ \varphi(S|m^t) &= 1 \text{ for all } m^t \text{ such that } m^t\neq G^t \text{ for some } t\leq\kappa,\end{aligned} \tag{24}$$

where (24) corresponds to off-equilibrium beliefs. It is easily seen that D3 is satisfied.

Let the value function for the sender as follows. For sender type $j\leq\kappa+1$,

$$\begin{aligned}V_S\left(m^t,j\right) &= (t-1)\,\delta_G \text{ for } m^t=G^t,\ t<\kappa+1. \\ V_S\left(m^t,j\right) &= -\delta\{N_G\left(m^t\right)+N_B\left(m^t\right)\} \text{ otherwise.}\end{aligned}$$

and for sender type $j > \kappa + 1$,

$$\begin{aligned} V_S\left(m^t, j\right) &= V - \delta\{N_G\left(m^t\right) + N_B\left(m^t\right)\} \text{ for } m^t \text{ such that } N_G\left(m^t\right) \geq \xi, \\ V_S\left(G^{\kappa+1}, j\right) &= V - \delta\left(\kappa + 1\right), \end{aligned}$$

$$\begin{aligned} & V_S\left(G^t, j\right) \\ = & \sum_{j=0}^{\kappa-t+1} \left(\frac{\delta}{V}\right)\left(1 - \frac{\delta}{V}\right)^j \left(V - \delta\left(t + j\right)\right) + \left(1 - \frac{\delta}{V}\right)^{\kappa-t+1} \left(V - \delta\left(\kappa+1\right)\right) \text{ for } t \leq \kappa + 1, \end{aligned}$$

and

$$V_S\left(m^t, j\right) = -\delta\{N_G\left(m^t\right) + N_B\left(m^t\right)\} \text{ otherwise.}$$

It is straightforward to verify that V_S satisfies condition for being a value function, given β.

Take a sender with $j \leq \kappa$ number of pieces of good evidence. From above, it follows that

$$V_S\left(\left(m^t, G\right), j\right) = \delta - \delta t = -\delta\left(t - 1\right) = V_S\left(\left(m^{k-1}, S\right), j\right).$$

Then he is indifferent between sending G and any S after $m^t = G^t$, with $t \leq j$, which shows that $\alpha\left(\cdot, \cdot, j\right)$ satisfies $D2$ for $j \leq \kappa$. For sender with $j > \kappa$ number of good aspects, on the other hand, we have

$$V_S\left(\left(G^t, G\right), j\right) > V_S\left(\left(G^t, S\right), j\right) = -\delta t > V_S\left(\left(G^t, B\right), j\right) = -\delta t - \delta,$$

for all $t \leq \kappa$, which shows that $\alpha\left(\cdot, \cdot, j\right)$ satisfies $D2$ for $j > \kappa$.

For the DM, let the value function be

$$\begin{aligned} V_{DM}\left(m^t\right) &= \sum_{j=0} B_j\left(G^t\right) \mathbb{E}[\theta|j] - \eta t \text{ for all } G^t, \\ V_{DM}\left(m^t\right) &= \eta\{N_G\left(m^t\right) + N_B\left(m^t\right)\} \text{ for all } m^t \text{ such that } m^t \neq G^t \text{ and } N_G\left(m^t\right) < \xi, \end{aligned}$$

and

$$\begin{aligned} V_{DM}\left(m^t\right) &= E[\theta|N] - \eta\{N_G\left(m^t\right) + N_B\left(m^t\right)\} \\ \text{for all } \left(m^t, m\right) \text{ such that } m^t &\neq G^t \text{ and } N_G\left(m^t\right) \geq \xi. \end{aligned}$$

It is straightforward to show that V_{DM} is a value function.

Think about the decision at the first period. The expected payoff from not continuing is $\max\{E[\theta], 0\}$. On the other hand, the expected payoff from entering period 1 and make decision at period 1 is given by (9). Hence from D1, $\beta\left(C, \varnothing\right) = 1$ is an optimal. Think about the decision after $m^t = G^t$ and $t \leq \kappa$. From the description of sender's strategy, the DM never receives B by choosing C and thus we can calculate

$$V_{DM}\left(G^t\right) = \mathbb{E}[U_{DM}(A, j, t)|G^t] - t\eta = \sum_{m \in \{G,B,S\}} \varphi(m|G^t) V_{DM}\left(G^t, m\right),$$

and hence $\beta(A, G^t)$ satisfies $D1$. It is easy to see that $D1$ is satisfied for the other cases as well, because in such cases we have $V_{DM}(m^t) > \sum_{m \in \{G,B,S\}} \varphi(m|G^t) V_{DM}(G^t, m)$ and

$$\beta\left(R, m^t\right) = 1 \text{ if } N_G\left(m^t\right) < \xi \text{ and } \beta\left(A, m^t\right) = 1 \text{ if } N_G\left(m^t\right) \geq \xi.$$

Q.E.D.

In order to prove Theorem 3, we first prove several lemmata. In the following, we fix an equilibrium $(\alpha, \beta, \varphi, B)$.

Lemma 3 *If* $(m^t, G) \in \Delta$ *and* $\beta(A, (m^t, G)) < 1$, *there must be some* $j \in P((m^t, G))$ *such that* $\max_{m \in M((m^t,G),j)} V_S((m^t, G, m), j) = -\delta\{N_G(m^t, G) + N_B(m^t, G)\}$.

Proof. It is seen from the proof of proposition 1. ■

Lemma 4 *For all* $n^s \in \Xi$ *and* $k^\tau \in \Xi$, *it holds that* $N_G(n^s) = N_G(k^\tau)$.

Proof. Take $n^\tau = (n_1, .., n_s) \in \Xi$ and $k^t = (k_1, .., k_\tau) \in \Xi$ and suppose that $N_G(n^s) > N_G(k^\tau)$. Let n^t be a longest sub-history of n^τ and k^τ such that $n^t = k^t$. Note that this can be $\varnothing$ if $n_1 \neq k_1$. We can show that $N_G(n^s) - 1 \in P(n^{s-1})$. To see this, note that from $N_G(n^s) > N_G(k^\tau)$, there must be some $n^{s'} \prec n^s$ such that $N_G(n^{s'}) = N_G(n^s) - 1$, which means that the DM knows that the sender has at least $N_G(n^s) - 1$ number of good evidence after history $n^{s'}$. Since $\beta(C, n^{s-1}) > 0$, the DM has to expect some type from $P(n^{s-1})$ chooses S, which can be only type $N_G(n^s) - 1$ sender. This implies that $V_S((n^t, n_{t+1}), N_G(n^s) - 1) \geq V_S((n^t, k_{t+1}), N_G(n^s) - 1)$. However, since the type $N_G(n^s) - 1$ sender can follow the path k^s instead and $\beta(A, k^s) = 1$, this implies that

$$\sum_{l=t}^{\tau-1} \left(1 - \beta\left(A, n^l\right)\right)^{t'-t} \beta\left(A, n^l\right)^l U_S\left(A, n^{t+1}\right) > \sum_{l=t}^{\tau-1} \left(1 - \beta\left(A, k^l\right)\right)^{t'-t} \beta\left(A, k^l\right)^l U_S\left(A, k^{t+1}\right)$$

, where each term is the expected payoff for the sender summed up before period $\tau - 1$. In other words, the path n^s has higher probability of acceptance than k^s does in the early period of the path. However, this implies that $V_S((n^t, n_{t+1}), N_G(k^{\tau-1}) - 1) > V_S((n^t, k_{t+1}), N_G(k^\tau) - 1)$ and hence $N_G(k^\tau) - 1 \notin P(k^{\tau-1})$. However this is a contradiction because we can show that $N_G(k^\tau) - 1 \in P(k^{\tau-1})$, from the same reasoning used above. ■

Take an equilibrium $(\alpha, \beta, \varphi, B)$. Let $\Delta_1 \in \Delta$ be a subset of Δ such that each $m \in \Delta_1$ contains only one G and as its final element, that is, m takes the form of $(G), (m_1, G), (m_1, m_2, G)$. Then we can show the following lemma. Define Δ_j in a similar manner.

Lemma 5 *Take an equilibrium. Then for all* $m \in \Delta_j$ *and* $m' \in \Delta_j$, $\beta(A, m) = \beta(A, m')$ *and* $\beta(C, m) = \beta(C, m')$ *for all* j.

Proof. We first show that for all $m \in \Delta_1$ and $m' \in \Delta_1$, $\beta(A, m) = \beta(A, m')$. To see this, suppose that there is some pair $m \in \Delta_1$ and $m' \in \Delta_1$ such that $\beta(A, m) > \beta(A, m')$. Then obviously, $1 \notin P(m')$. Also because $\beta(A, m') < 1$, from Lemma 3 there must be some $j \in P(m')$ such that $V_S((m', m), j) = -\delta\{N_G(m') + N_B(m')\}$. However, then we have $V_S(m, j) > V_S(m', j)$, which contradicts $j \in P(m')$. Then we can prove that $\beta(A, m) = \beta(A, m')$ for $m \in \Delta_j$ and $m' \in \Delta_j$ inductively. We can also prove that $\beta(C, m) = \beta(C, m')$ for all $m \in \Delta_j$ and $m' \in \Delta_j$ in a similar manner. ■

Lemma 6 *For all $m \in \Delta$ and $m' \in \Delta$ such that $\beta(A, m) = \beta(A, m') = 1$, $N_B(m) = N_B(m')$.*

Proof. Suppose that $m \in \Delta$ and $m' \in \Delta$, $\beta(A, m) = \beta(A, m') = 1$ but $N_B(m) > N_B(m')$. Then it is immediately seen that from Lemma 5, for all $j \in P(m)$, we have $V_S(m_1', j) > V_S(m_1, j)$, where $m_1' \in \Delta_1$, $m_1' \prec m'$, $m_1 \in \Delta_1$, $m_1 \prec m$ (note that history m' reaches the node of sure acceptance faster than m does), which contradicts with $m \in \Delta$. ■

This lemma allows us to define the function that gives the number of pieces of good and bad evidence to persuade the DM for a given equilibrium, which we denote by $N_G(e)$ and $N_B(e)$.

Proof of Theorem 3 is divided into three parts.

Lemma 7 *For all $e \in \mathcal{P}(\eta, \eta_S, \delta)$, it holds that $\beta(A, (m^t, G)) = 1$ or $\beta(A, (m^t, G)) = \delta/V$ for all $(m^t, G) \in \Delta$.*

Proof. Suppose that the set of message history $M^+ = \{m^t | (m^\tau, G) \in \Delta$ such that $\beta(A, (m^\tau, G)) \in (\delta/V, 1)$ is non-empty. Also, let M^{++} be the set of the smallest elements of M^+ with respect to the order of $\prec$, that is, if $m^s \in M^{++}$ there is no $m^t \in M^+$ such that $m^t \prec m^s$. Note that from Lemma 6, $N_G(m^s) = N_G\left(m^{s'}\right) = \gamma$ for some γ for all $m^s \in M^{++}$ and $m^{s'} \in M^{++}$. Also, for all $m^\tau \in \Xi$ we can find a $m^t \in M^{++}$ such that $m^t \prec m^\tau$.

In the equilibrium e, for every sender type j such that $N - \gamma > j \geq \gamma$, there is a $m_1 \in M(\varnothing, j)$ such that $V_s(m_1, j) > 0$, that is, every sender types from $\{N_G(M^{++}), .., N - N_B(M^{++}) - 1\}$ can get strictly positive payoffs by following a message path from M^{++} (note that from proposition 1, each time a sender communicates an aspect, he is accepted with a probability of as high as η/V, which is enough to recover the cost of communication one time).

In this case, we can construct equilibrium $\widehat{e} = \left(\widehat{\alpha}, \widehat{\beta}, \widehat{B}, \widehat{\varphi}\right)$ in an way that $\left(\widehat{\alpha}, \widehat{\beta}, \widehat{B}, \widehat{\varphi}\right) = (\alpha, \beta, B, \varphi)$ except $\widehat{\beta}(A, m^s) = 1$ for all $m^s \in M^{++}$. To see that this is an equilibrium, first note that

$$
\begin{aligned}
V_S^e(m, m^s, x) &= V_S^{\widehat{e}}(m, m^s, x) \text{ for all } m^s \preceq m^\tau \text{ such that } m^\tau \notin \Xi^{\widehat{e}} = M^{++} \\
V_S^e(m, m^s, x) &= V_S^{\widehat{e}}(m, m^s, x) \text{ for all } m^s \preceq m^\tau \text{ such that } m^\tau \in \Xi^{\widehat{e}} = M^{++},
\end{aligned}
$$

and $V_S^{\widehat{e}}(m, m^s, j) = V_S^{\widehat{e}}(m, m^s, j)$ for all m^s, m_{s+1}, and m_{s+1}' such that there exist m^τ and $m^{\tau'}$ such that $(m^s, m_{s+1}) \preceq m^\tau \in \Xi^{\widehat{e}}$ and $\left(m^s, m_{s+1}'\right) \preceq m^{\tau'} \in \Xi^{\widehat{e}}$. Those implies that $\widehat{\alpha}$ satisfies D-2 given $\widehat{\beta}$, from the fact that α satisfies D-2 given β. On the other hand, for all m^t we have $V_{DM}^{\widehat{e}}(m^t) \leq V_{DM}^e(m^t)$ for all m^t such that there is $m^\tau \in \Xi^{\widehat{e}}$ and $m^t \prec m^\tau$. Also, $V_{DM}^{\widehat{e}}(m^t) = V_{DM}^e(m^t)$ if there is not such m^τ, and hence $\widehat{\beta}$ satisfies D-1 given $\left(\widehat{\varphi}, \widehat{B}\right)$, from the fact that β satisfies D-1 given (φ, B). Hence $\widehat{e}$ is an equilibrium. We can also see that $\widehat{e}$ Pareto dominates e since it has strictly higher payoff for sender types $\{N_G(M^{++}), .., N - N_B(M^{++}) - 1\}$ while giving exactly the same payoff for other types of sender and the DM. ■

Lemma 8 *For all $e \in \mathcal{P}(\eta, \eta_S, \delta)$, it holds that $\beta(A, (m^t, G)) = 1$ or $\beta(A, (m^t, G)) = \delta/V$ for all $(m^t, G) \in \Delta$.*

Proof. From Lemma 8, we assume that $\beta(A,(m^t,G)) \in \{1,\delta/V\}$ for all $(m^t,G) \in \Delta$ holds in any equilibrium we consider. Take an equilibrium e such that $\beta(R,(m^t,S)) < 1$ for some $(m^t,S) \in \Delta$. Let Ψ be a set of such m^t. To get a contradiction, m^t is not empty. From Lemma 4, $N_G(m^\tau) = N_G\left(m^{\tau'}\right) = N_G(e)$ and $N_B(m^\tau) = N_B\left(m^{\tau'}\right) = N_B(e)$ for all $m^\tau \in \Xi^e$ and $m^{\tau'} \in \Xi^e$. From the equilibrium condition, for all $m^{t+1} = (m^t, m_{t+1}) \in \Delta$ such that $m_{t+1} \in \{G,B\}$, we have

$$\begin{aligned} & -\sum_{j\geq 1}^{N}\{1-\sum_{m\in\{G,B\}}\alpha\left(m,m^{t+1},j\right)\}\alpha\left(m^{t+1},j\right)f(j)\left(\mathbb{E}[\theta|j]+\eta_S\right) \\ \geq\ & \eta\sum_{n\geq 1}^{N}\sum_{m\in\{G,B\}}\alpha\left(m,m^{t+1},j\right)f(j), \end{aligned} \tag{25}$$

which follows from $\sum_{j=0}^{N} B_j(m^t)\, U_{DM}(A,j,m^t) = \sum_{m\in M}\varphi(m|m^t)V_{DM}(m_t,m)$ and Proposition 1. Note that this holds with equality when $\beta(R,(m^{t+1},S)) = 1$ and strict inequality otherwise, and from the maintained assumption we have (25) with strictly inequality for at least one $m^t \in \Delta$. Let $\Gamma(e)$ be a set of message histories such that their last element is G or B and rest of elements are S, that is, it is the set of message history that an piece of evidence is communicated for the first time.

Then, we can construct following equilibrium $\widehat{e}$: there is only one acceptance history $m^\tau = (B,..,B,G,..,G)$, that is, $\widehat{\beta}(R,m^t) = 1$ for all $m^t \not\preceq m^\tau$, $\widehat{\beta}(A,m^\tau) = 1$. The sender's strategy $\widehat{\alpha}$ satisfies (8) with inequality for all $m^t \prec m^\tau$. Then, it is possible to choose $\widehat{\alpha}$ in such a way that $\widehat{\alpha}(B,\varnothing,j) \leq \sum_{m\in\Gamma(e)}\alpha(m,j)$ for all $j \leq N_G(m^\tau)$ and at least one strictly inequality. However this implies that

$$\begin{aligned} V_{DM}^{\widehat{e}}(\varnothing) &= \sum_{n\geq 1}^{N}\widehat{\alpha}(B,\varnothing,j)\left(\mathbb{E}[\theta|j]-\eta\right) - \eta_S\sum_{n\geq 1}^{N}(1-\widehat{\alpha}(B,\varnothing,j)) \\ &> \sum_{n\geq 1}^{N}\sum_{m\in\Gamma(e)}\alpha(m,j)\left(\mathbb{E}[\theta|j]-\eta\right) - \eta_S\sum_{n\geq 1}^{N}(1-\sum_{m\in\Gamma(e)}\alpha(m,j)) \geq V_{DM}^{e}(\varnothing). \end{aligned}$$

■

Lemma 9 *If there is an equilibrium e such that $(m^t,B) \in \Delta$ for some m^t, then there is a benchmark strategy equilibrium e' that gives the DM strictly higher payoff.*

Proof. Take an equilibrium $e = (\alpha,\beta,B,\varphi)$ that involves communicating a piece of bad aspect one time. From Lemma 6, all the equilibrium acceptance path involves at least one time communication of a piece of bad evidence, which implies that type N sender has to drop before reaching "acceptance for sure" nodes. Without loss of generality, assume that $m^\tau = (m_1,..,m^\tau) \in \Xi$ contains no S, that is, $m_t \in \{G,B\}$ and thus $N_G(e)+1 = \tau$. Moreover, from Lemma 7, we can assume that for all $(m^t, G) \in \Delta,\ \beta(A, (m^t, G)) \in \{\eta/V, 1\}$. Then,

$$\begin{aligned} V_{DM}(m_1) &= \sum_{j\geq 1}^{N} \alpha(m_1, \varnothing, j) f(j) (\mathbb{E}[\theta|j] + \eta) - \eta_S \sum_{j\geq 1}^{N} \{1 - \alpha(m_1, \varnothing, j)\} f(j) = \\ & \eta \sum_{t=1}^{2} \sum_{j\geq 1}^{N} \Pi_{s=1}^{t} \alpha\left(m_s, m^{s-1}, j\right) f(j) + \sum_{j\geq 1}^{N} \Pi_{s=1}^{k} \alpha\left(m_s, m^{s-1}, j\right) f(j) \mathbb{E}[\theta|j] \\ &= \sum_{j\geq \kappa}^{N-1} f(j) \mathbb{E}[\theta|j] - \eta \sum_{t=1}^{\tau} \sum_{j\geq 1}^{N} \Pi_{s=1}^{\tau} \alpha\left(m_s, m^{s-1}, j\right) f(j), \end{aligned} \tag{26}$$

where we used the relation

$$-\sum_{j\geq 1}^{N} \{1-\alpha\left(m_{t+1}, m^t, j\right)\} \Pi_{s=1}^{t} \alpha\left(m_s, m^{s-1}, j\right) f(j) (\mathbb{E}[\theta|j] + \eta_S) = \eta \sum_{j\geq 1}^{N} \Pi_{s=1}^{t+1} \alpha\left(m_s, m^{s-1}, j\right) f(j), \tag{27}$$

for all m_{t+1}. Since type N sender has to drop at some period and $V_{DM}(m_1) \geq 0$, this implies that

$$-\frac{\sum_{j\geq\kappa}^{N-1} f(j) (\mathbb{E}[\theta|j] + \eta_S)}{\sum_{j\geq\kappa}^{N-1} f(j)} \geq \eta \frac{\sum_{t=1}^{\tau} \sum_{j\geq 1}^{N} \Pi_{s=1}^{t} \alpha(m_s, m^{s-1}, j) f(j)}{\sum_{j\geq\kappa}^{N-1} f(j)} \tag{28}$$

We will first show that there exists a benchmark strategy equilibrium such that it gives strictly higher payoff to the DM than equilibrium e does. We moreover assume that $m_1 = B$.

Take a benchmark strategy equilibrium $e' = (\alpha', \beta', \varphi', B')$ that maximizes $N_G(e')$. First, suppose that we have a $N_G(e') \geq N_G(e) - 1$. Define

$$\begin{aligned} \Delta\alpha(t) &= \Pi_{s=1}^{t-1} \alpha'\left(m_s, m^{s-1}, j\right) \left(1 - \alpha'\left(m_t, m^{t-1}, j\right)\right) f(j) \\ & \quad -\Pi_{s=1}^{t-1} \alpha\left(m_s, m^{s-1}, j\right) \left(1 - \alpha'\left(m_t, m^{t-1}, j\right)\right) f(j) \\ \text{and } \Delta E(j) &= \Pi_{s=1}^{t-1} \alpha'\left(m_s, m^{s-1}, j\right) \left(1 - \alpha'\left(m_t, m^{t-1}, j\right)\right) \mathbb{E}[\theta|j] \\ & \quad -\Pi_{s=1}^{t-1} \alpha\left(m_s, m^{s-1}, j\right) \left(1 - \alpha'\left(m_t, m^{t-1}, j\right)\right) \mathbb{E}[\theta|j] \end{aligned}$$

If $N_G(e') = N_G(e) - 1$, because we cannot construct an equilibrium in an way that k aspects are communicated, we must have $-f(\kappa - 1) \{\mathbb{E}[\theta|\kappa - 1] + \eta_S\} < \eta \sum_{j\geq\kappa}^{N} f(j)$. To see this, note that If $-f(N_G(e)) \{\mathbb{E}[\theta|N_G(e)] + \eta_S) \geq \eta \sum_{j\geq N_G(e)+1}^{N}$, by the fact that we can construct an equilibrium e' with $N_G(e') = N_G(e)$ implies that we can construct another equilibrium e' with $N_G(e') = N_G(e)$ by setting $\alpha'\left(G, G^{k-1}, N_G(e)\right) = -\eta \sum_{j\geq N_G(e)+1}^{N} / \{\mathbb{E}[\theta|N_G(e)] + \eta\}$, and $\alpha' \leq \alpha(G, G^{n-1}, j)$ for all n, j, which is a contradiction. Then because we have $\Pi_{s=1}^{\tau} \alpha(m_s, m^{s-1}, j) f(N_G(e) - 1) \{\mathbb{E}[\theta|N_G(e) - 1] + \eta_S\} = \eta \sum_{j\geq\kappa}^{N-1} f(j)$ from the construction of equilibrium e, we have

$$\Delta E(\tau) < \eta f(N) \text{ and } \Delta\alpha(\tau) < \frac{1}{E(N_G(e) - 1)} \eta f(N) \tag{29}$$

On the other hand, if $N_G(e') \geq \kappa$, (29) follows immediately. Those in turn imply that

$$\Delta E(\tau - 1) < \eta f(N) + \Delta\alpha(\tau) \eta \text{ and } \Delta\alpha(\tau - 1) < \frac{1}{E(N_G(e) - 1)} \Delta\mathbb{E}(\tau - 1).$$

By continuing this, we will get

$$\Delta E\left(t\right) < \eta f\left(N\right) + \sum_{s=t+1}^{\tau} \Delta\alpha\left(s\right) \text{ for all } j > 1.$$

Next we show that

$$\Delta E\left(t\right) < \eta f\left(N\right) \frac{\sum_{j\geq 1}^{N} \Pi_{s=t}^{\tau} \alpha\left(m_s, m^{s-1}, j\right) f\left(j\right)}{\sum_{j\geq\kappa}^{N-1} f\left(j\right)}. \tag{30}$$

To see this, note that

$$\begin{aligned}
& \frac{\sum_{j\geq 1}^{N} \Pi_{s=1}^{t} \alpha\left(m_s, m^{s-1}, j\right) f\left(j\right)}{\sum_{j\geq\kappa}^{N-1} f\left(j\right)} \\
= & \frac{\sum_{s=t}^{\tau} \sum_{j\geq 1}^{N} \Pi_{n=t}^{s} \left(1-\alpha\left(m_{s+1}, m^{s}, j\right)\right) \alpha\left(m_n, m^{n-1}, j\right) f\left(j\right) + \sum_{j\geq\kappa}^{N-1} f\left(j\right)}{\sum_{j\geq\kappa}^{N-1} f\left(j\right)} \\
= & \frac{\Gamma\left(t+1\right) + ... + \Gamma\left(\tau-1\right) + \Gamma\left(\tau\right)}{\sum_{j\geq\kappa}^{N-1} f\left(j\right)},
\end{aligned}$$

where we defined $\Gamma\left(r\right) = \sum_{s=1}^{r} \sum_{j\geq 1}^{N} \Pi_{s=1}^{r} \left(1-\alpha\left(m_r, m^{r-1}, j\right)\right) \alpha\left(m_{s-1}, m^{s}, j\right) f\left(j\right).$

On the other hand we can show that

$$\begin{aligned}
\Delta E\left(\tau\right) &< \eta f\left(N\right) = \eta f\left(N\right) \frac{\sum_{j\geq\kappa}^{N-1} f\left(j\right)}{\sum_{j\geq\kappa}^{N-1} f\left(j\right)} = \eta f\left(N\right) \Gamma\left(\tau\right), \\
\Delta E\left(\tau-1\right) &< \eta f\left(N\right) + \Delta\alpha\left(\tau\right)\eta = \eta f\left(N\right)\Gamma\left(\tau\right) + \eta f\left(N\right) \frac{\eta}{\mathbb{E}\left(\kappa-1\right)} \\
&\leq \eta f\left(N\right)\Gamma\left(\tau\right) + \eta f\left(N\right) \frac{\Pi_{s=t}^{\tau-1} \alpha\left(m_s, m^{s-1}, \kappa-1\right) f\left(\kappa-1\right)}{\sum_{j\geq\kappa}^{N-1} f\left(j\right)} \\
&= \eta f\left(N\right)\Gamma\left(\tau\right) + \eta f\left(N\right)\Gamma\left(\tau-1\right),
\end{aligned}$$

and more generally, $\Delta E\left(r\right) < \eta f\left(N\right) \sum_{j\geq r}^{\tau} \Gamma\left(\tau\right)$, which implies

$$\Delta E\left(r\right) < \frac{\sum_{j\geq 1}^{N} \Pi_{s=1}^{r+1} \alpha\left(m_s, m^{s-1}, j\right) f\left(j\right)}{\sum_{j\geq\kappa}^{N-1} f\left(j\right)} \text{ for all } r \in \{1, .., \kappa-1\}$$

Obliviously, we have

$$0 \leq V_{DM}^{e}\left(\varnothing\right) - V_{DM}^{e'}\left(\varnothing\right) < \sum_{j=1}^{\kappa-1} \Delta E\left(\tau\right) - \mathbb{E}\left(\theta|N\right)$$

and thus (30) implies

$$\mathbb{E}\left(\theta|N\right) > \frac{\sum_{t=1}^{\tau} \sum_{j\geq 1}^{N} \Pi_{s=1}^{t} \alpha\left(m_s, m^{s-1}, j\right) f\left(j\right)}{\sum_{j\geq\kappa}^{N-1} f\left(j\right)},$$

which contradicts with (28) since $\mathbb{E}[\theta|j]$ is increasing with j.

Next, think of the case in which we have a benchmark strategy equilibrium $e' = (\alpha', \beta', \varphi', B')$ such that $N_G(e') = \kappa - 2$ but not $\kappa - 1$. Then the fact that we cannot construct an equilibrium in an way that $k-1$ aspects are communicated implies

$$-f(\kappa-1)\mathbb{E}[\theta|\kappa-1] - f(\kappa-2)\mathbb{E}[\theta|\kappa-2] < 2\eta\sum_{j\geq\kappa}^{N} f(j) + \eta^2 \frac{\sum_{j\geq\kappa}^{N} f(j)}{\mathbb{E}[\theta|\kappa-1]}.$$

Then by using the same argument we can get a contradiction. Other cases can be treated similarly. ■

Proposition 3 follows immediately from Lemma 7, 8, and 9. *Q.E.D.*

8.4 Proof for Section 6

Proof of Proposition 4: Fix model's parameter values (η, η_S, δ). In a benchmark strategy equilibrium, the sender's strategy α can be seen as a element of $[0,1]^{2(\xi-1)}$ (each represents the probability of communicating G), since $\alpha(G, G^t, j) = 1$ for all $j \geq \xi$ and $t \leq \xi$. Let $\mathcal{E} \subset [0,1]^{2(\xi-1)}$ be the set of the sender's strategy that is supported as an equilibrium, and let $\mathcal{E}_\lambda \subset \mathcal{E}$ for $\lambda \in \{1, ..., \xi\}$ be a subset of the sender's strategy that is supported by an equilibrium e such that $N_G(e) = \lambda$. We first show that set $\mathcal{E}_\lambda$ is closed in the usual sense of Euclidean topology. However this is easy because if we take a sequence $\{\alpha^n\}_{n=1}^{\infty}$ from $\mathcal{E}$ that converges to α, it holds that

$$-\sum_{j\geq x}^{N}\{1-\alpha^n\left(G, G^t, j\right)\}\Pi_{s=1}^{t}\alpha^n\left(G, G^{s-1}, j\right) f(j)\left(\mathbb{E}[\theta|j] + \eta_S\right) = \eta\sum_{j\geq x}^{N}\Pi_{s=1}^{t+1}\alpha^n\left(G, G^{s-1}, j\right) f(j),$$

for all n and $t \leq \lambda$, which implies that the same condition holds for α from $\lim_{n\to\infty}\alpha^n \to \alpha$. Hence $\alpha \in \mathcal{E}_\lambda$ and $\mathcal{E}_\lambda$ is closed. Then $V_{DM}(\varnothing)$, calculated as $\sum_{j\geq 1}^{N}\alpha(G, \varnothing, j) f(j)(\mathbb{E}[\theta|j] - \eta)$, is a continuous function on $\mathcal{E}_\lambda$, which is closed and bounded, and hence has a maximum point in $\mathcal{E}_\lambda$. Then $V_{DM}(\varnothing)$ has the maximum on $\mathcal{E}$, which is a finite union of $\mathcal{E}_\lambda$.

We next prove the uniqueness. Towards this end, suppose that we have two different equilibria e and e' such that $V_{DM}^{e}(e) = V_{DM}^{e}(e)$ and those maximize the DM's expected payoff (best equilibria). From Theorem 4, which will be proved below, $N_G(e) = N_G(e') = \lambda$ and $\alpha(G, G^t, j) = \alpha'(G, G^t, j) = 1$ for all $t \leq N_G(e)$ and $j \geq N_G(e)$. Moreover, since (11) must hold at period $\lambda - 1$, we must have $\alpha\left(G^{\lambda-1}, \lambda-1\right) = \alpha'\left(G^{\lambda-1}, \lambda-1\right)$. Let $h < \lambda - 1$ be the biggest t such that $\alpha(G^t, j) \neq \alpha'(G^t, j)$ for some $j \leq h$, and let l be the biggest $l \leq h$ such that $\alpha\left(G^h, j\right) \neq \alpha'\left(G^h, j\right)$. Without loss of generality, let $1 \geq \alpha\left(G^h, j\right) > \alpha'\left(G^h, j\right)$. Since we have (8) for period h, there must be some q such that $\alpha\left(G^h, q\right) < \alpha'\left(G^h, q\right) \leq 1$. Then we can find a pair of strictly positive numbers $\varepsilon < \alpha'\left(G^h, q\right) - \alpha\left(G^h, q\right)$, $\delta < \alpha\left(G^h, l\right) - \alpha'\left(G^h, l\right)$, and ϵ such that

$$\begin{aligned}
& \{1-\alpha\left(G, G^{h-1}, q\right)\}\Pi_{s=1}^{h-2}\alpha^n\left(G, G^{s-1}, q\right) f(q)\{\mathbb{E}[\theta|q] + \eta_S\} \\
& +\{1-\alpha\left(G, G^{h-1}, l\right)\}\Pi_{s=1}^{h-2}\alpha^n\left(G, G^{s-1}, l\right) f(l)\{\mathbb{E}[\theta|l] + \eta_S\} \\
= \; & \{1-\alpha\left(G, G^{h-1}, q\right) - \varepsilon\}\left(\alpha\left(G^{h-1}, q\right) - \epsilon\right) f(q)\{\mathbb{E}[\theta|q] + \eta_S\} \\
& +\{1-\alpha\left(G, G^{h-1}, l\right) + \delta\}\left(\alpha\left(G^{h-1}, l\right) - \epsilon\right) f(l)\{\mathbb{E}[\theta|l] + \eta_S\},
\end{aligned}$$

because $\mathbb{E}[\theta|j]$ is strictly increasing with j. Then think of the following strategy after after period $h-1:$ $\widehat{\alpha}(G^t, j) = \alpha(G^t, j)$ for all $j \neq l, q$, and

$$\begin{aligned}
\widehat{\alpha}\left(G, G^{h-1}, q\right) &= \alpha\left(G, G^{h-1}, q\right) + \varepsilon \text{ and } \widehat{\alpha}\left(G, G^{h-1}, q\right) = \alpha\left(G, G^{h-1}, q\right) - \delta; \\
\Pi_{s=1}^{h-2}\widehat{\alpha}\left(G, G^{s-1}, q\right) &= \alpha\left(G^{h-1}, q\right) - \epsilon \text{ and } \Pi_{s=1}^{h-2}\widehat{\alpha}\left(G, G^{s-1}, l\right) = \alpha\left(G, G^{h-1}, l\right) - \epsilon.
\end{aligned}$$

Then obviously, $\sum_j \Pi_{s=1}^{h-2}\widehat{\alpha}(G, G^{s-1}, j) f(j) < \sum_j \Pi_{s=1}^{h-2}\alpha(G, G^{s-1}, j) f(j)$, and equilibrium condition (8) is satisfied after period $h-1$. Then, by taking the same step as in the proof of Theorem 4, we can construct an equilibrium that can support such strategy after $h-1$ in an way such that $\widehat{\alpha}(G, \varnothing, j) f(j) \leq \alpha(G, \varnothing, j)$ for all j holds with at least one strict inequality. Obviously, such an equilibrium attains strictly higher $V_{DM}(\varnothing)$ than e does and it is a contradiction. *Q.E.D.*

Proof of Theorem 4: To get a contradiction, suppose that equilibrium e is the best equilibrium for the DM, but there is another equilibrium $\widehat{e}$ such that $N_G(\widehat{e}) > N_G(e)$. Obviously, $N_G(\widehat{e}) \leq \xi$. Then from the fact that e being an equilibrium, we have (8) for all $t < N_G(e)$ and $\Pi_{s=1}^{t}\alpha(G, G^{s-1}, j) = 1$ for all $j \geq N_G(e)$. On the other hand, since $N_G(\widehat{e}) > N_G(e)$, we have

$$\begin{aligned}
& -\sum_{j\geq 1}^{N}\{1 - \widehat{\alpha}\left(G, G^t, j\right)\}\Pi_{s=N_G(e)}^{t}\widehat{\alpha}\left(G, G^{s-1}, j\right) f(j)\left(\mathbb{E}[\theta|j] + \eta_S\right) \\
= & \ \eta\sum_{j\geq 1}^{N}\Pi_{s=N_G(e)}^{t+1}\widehat{\alpha}\left(G, G^{s-1}, j\right) f(j) > 0,
\end{aligned}$$

for all $t < N_G(\widehat{e})$ and

$$\Pi_{s=1}^{t}\widehat{\alpha}\left(G, G^{s-1}, j\right) = 1 \text{ for all } j \geq N_G(\widehat{e}) \text{ and } t \leq N_G(\widehat{e}).$$

Then from the assumption 1, it holds that $\Pi_{s=1}^{N_G(e')}\widehat{\alpha}(G, G^{s-1}, N_G(e') - 1) < 1$. Hence

$$-\eta\sum_{j\geq N_G(e)}^{N}\Pi_{s=1}^{N_G(e)}\alpha\left(G, G^{s-1}, j\right) f(j) - \eta\sum_{j\geq N_G(e)}^{N}\Pi_{s=1}^{N_G(e)}\widehat{\alpha}\left(G, G^{s-1}, j\right) f(j) > 0. \tag{31}$$

Then we can construct a benchmark strategy equilibrium $e' = (\alpha', \beta', \varphi', B')$ in the following way such that

$$\alpha'(,,j) = \widehat{\alpha}(,,j) \text{ for all } j \geq N_G(\widehat{e}) - 1, \ \beta' = \widehat{\beta},$$

(8) for all $t < N_G(\widehat{e})$, and

$$\alpha'\left(G, G^t, j\right) \leq \alpha\left(G, G^t, j\right) \text{ for all } j < N_G(\widehat{e}) \text{ and } t < N_G(\widehat{e}),$$

which is possible from (31). Then it follows that

$$\begin{aligned}
V_{DM}^{e'}(\varnothing) &= \sum_{j\neq N_G(e')-1}^{N} \alpha'(G,\varnothing,j) f(j)(\mathbb{E}[\theta|j]-\eta) \\
&\quad +\alpha'(G,\varnothing,j) f(N_G(e')-1)(\mathbb{E}[\theta|N_G(e')-1]-\eta) \\
&\quad -\eta_S[\sum_{j\neq N_G(e')-1}\{1-\alpha'(G,\varnothing,j)\}f(j)+\{1-\alpha'(G,\varnothing,N_G(e')-1)\}f(N_G(e')-1)] \\
&> \sum_{j\neq N_G(e')-1}^{N} \alpha(G,\varnothing,j) f(j)(\mathbb{E}[\theta|j]-\eta) \\
&\quad +\alpha(G,\varnothing,j) f(N_G(e')-1)(\mathbb{E}[\theta|N_G(e')-1]-\eta) \\
&\quad -\eta_S[\sum_{j\neq N_G(e')-1}^{N}\{1-\alpha(G,\varnothing,j)\}f(j)+\{1-\alpha(G,\varnothing,N_G(e')-1)\}f(N_G(e')-1)] \\
&= V_{DM}^{e}(\varnothing),
\end{aligned}$$

which contradicts e being the best equilibrium for the DM. *Q.E.D.*

Proof of Theorem 5: That the strategies constructed by the procedure is an equilibrium follows from Theorem 4. Let λ be the biggest j such that $c_j < 1$ in the above process. Then we have $\Gamma(j) < \eta$ for all $j > \lambda$. Suppose that $\widehat{e} = (\widehat{\alpha}, \widehat{\beta}, \widehat{\varphi}, \widehat{B})$ is the best equilibrium. From Lemma 9 and Theorem 5, it must hold that $\Pi_{s=1}^{\lambda}\widehat{\alpha}(G, G^{s-1}, \lambda) = c_\lambda$. Suppose that $\Pi_{s=1}^{\lambda-1}\widehat{\alpha}(G, G^{s-1}, \lambda-1) \neq c_{\lambda-1}$. because if $\Pi_{s=1}^{\lambda-1}\widehat{\alpha}(G, G^{s-1}, \lambda-1) > c_{\lambda-1}$ the (8) cannot be satisfied at period $\gamma-2$, it must hold that $\Pi_{s=1}^{\lambda-1}\widehat{\alpha}(G, G^{s-1}, \lambda-1) < c_{\lambda-1}$. Then from the choice of $c_{\lambda-1}$, we have to have $1-\widehat{\alpha}\left(G, G^{\lambda-2}, \lambda\right) > 0$ so that

$$\begin{aligned}
&-\{1-\widehat{\alpha}\left(G,G^{\lambda-2},\lambda-1\right)\}\Pi_{s=1}^{\lambda-2}\widehat{\alpha}\left(G,G^{s-1},\lambda-1\right) f(\lambda-1)(\mathbb{E}[\theta|\lambda-1]+\eta_S) \qquad (32)\\
&-\{1-\widehat{\alpha}\left(G,G^{\lambda-2},\lambda\right)\}\Pi_{s=1}^{\lambda-2}\widehat{\alpha}\left(G,G^{s-1},\lambda\right) f(\lambda)(\mathbb{E}[\theta|\lambda]+\eta_S) \\
=\ & \eta\sum_{j\geq\lambda-1}^{N}\Pi_{s=1}^{\lambda-1}\widehat{\alpha}\left(G,G^{s-1},j\right) f(j) \\
=\ & -c_{\lambda-1}f(\lambda-1)(\mathbb{E}[\theta|\lambda-1]+\eta_S)-c_\lambda f(\lambda)(\mathbb{E}[\theta|\lambda]+\eta_S).
\end{aligned}$$

Because $|\mathbb{E}[\theta|\lambda-1]| > |\mathbb{E}[\theta|\lambda]|$, (32) implies that

$$\begin{aligned}
& \eta\sum_{j\geq\lambda-1}^{N}\Pi_{s=1}^{\lambda-1}\widehat{\alpha}\left(G,G^{s-1},j\right) f(j) \\
=\ & \eta\{\Pi_{s=1}^{\lambda-1}\widehat{\alpha}\left(G,G^{s-1},\lambda-1\right) f(\lambda-1)+\Pi_{s=1}^{\lambda-1}\widehat{\alpha}\left(G,G^{s-1},\lambda\right) f(\lambda)+\sum_{j\geq\lambda+1}^{N} f(j)\} \\
>\ & \eta\{c_{\lambda-1}f(\lambda-1)+c_\lambda f(\lambda)+\sum_{j\geq\lambda+1}^{N} f(j)\},
\end{aligned}$$

which implies in the equilibrium condition (8) at period $\lambda-3$, the right hand side is strictly bigger in equilibrium $\widehat{e}$ than in the equilibrium generated by the procedure. However, then it is possible to construct another equilibrium $e' = (\alpha', \beta', \varphi', B')$ by letting $\Pi_{s=1}^{\lambda-1}\alpha'(G, G^{s-1}, \lambda-1) = c_{\lambda-1}$, $\Pi_{s=1}^{\lambda}\alpha'(G, G^{s-1}, \lambda) = c_{\lambda}$, and

$$\eta \sum_{j\geq\lambda-1}^{N} \Pi_{s=1}^{t}\alpha'\left(G, G^{s-1}, j\right) f(j) < \eta \sum_{j\geq\lambda-1}^{N} \Pi_{s=1}^{t}\widehat{\alpha}\left(G, G^{s-1}, j\right) f(j) \text{ for all } t \in \{2, .., \lambda-2\}.$$

Then obviously,

$$V_{DM}^{e'}(\varnothing) = \sum_{j}^{N} \Pi_{s=1}^{t}\alpha'(G, \varnothing, j) f(j) (\mathbb{E}[\theta|j] - \eta) - \eta_S \sum_{j}^{N} \Pi_{s=1}^{t}\{1 - \alpha'(G, \varnothing, j)\} f(j)$$

$$> \sum_{j}^{N} \Pi_{s=1}^{t}\widehat{\alpha}(G, \varnothing, j) f(j) (\mathbb{E}[\theta|j] - \eta) - \eta_S \sum_{j}^{N} \Pi_{s=1}^{t}\{1 - \widehat{\alpha}(G, \varnothing, j)\} f(j) = V_{DM}^{\widehat{e}}(\varnothing),$$

which contradict with the fact that $\widehat{e}$ is the best equilibrium, from which $\Pi_{s=1}^{\lambda-1}\widehat{\alpha}(G, G^{s-1}, \lambda-1) = c_{\lambda-1}$ follows. Following the same procedure backward, we can eventually get $\Pi_{s=1}^{j}\widehat{\alpha}(G, G^{s-1}, j) = c_j$ for all j, which shows that our procedure generates the best equilibrium. *Q.E.D.*

8.5 Proof for Section 7

Proof of Theorem 8: The first statement is straightforward. In the benchmark strategy equilibrium e, we have $V_S(\varnothing, j) = 0$ for $j < N_G(e)$ and

$$\begin{aligned}
V_S(\varnothing, j) &= \sum_{s\in\{1,..,N_G(e)-1\}} (V - s\delta)(1 - \delta/V)^{s-1} \delta/V + (V - N_G(e)\delta)(1 - \delta/V)^{N_G(e)-1} \\
&= \sum_{s\in\{1,..,N_G(e)-1\}} (V - s\delta)(1 - \delta/V)^{s-1} \delta/V - (1 - \delta/V)^{N_G(e)-1} N_G(e) \\
&\quad + (1 - \delta/V)^{N_G(e)-1} N_G(e) + (V - N_G(e)\delta)(1 - \delta/V)^{N_G(e)-1} \\
&= V_S(\varnothing, N_G(e) - 1) + (1 - \delta/V)^{N_G(e)-1} N_G(e) + (V - N_G(e)\delta)(1 - \delta/V)^{N_G(e)-1} \\
&= (1 - \delta/V)^{N_G(e)-1} N_G(e) + (V - N_G(e)\delta)(1 - \delta/V)^{N_G(e)-1},
\end{aligned}$$

for $j \geq N_G(e)$, where we used the fact $V_S(\varnothing, N_G(e) - 1) = 0$. Then obviously $\frac{\partial V_S(\varnothing,j)}{\partial\delta} < 0$, which implies the result. *Q.E.D.*

Proof of $\sum_{j=1}^{N_G(e)-1} \frac{\partial\alpha(G,\varnothing,j)}{\partial\eta} f(j)(\mathbb{E}[\theta|1] - \eta) \leq 0$.

To get a contradiction, suppose that $\sum_{j=1}^{N_G(e)-1} \frac{\partial\alpha(G,\varnothing,j)}{\partial\eta}_{|\eta=\eta'} f(j)(\mathbb{E}[\theta|1] - \eta') > 0$ for some η'. Then, we have $\eta'' > \eta'$ sufficiently close to η' and $\sum_{j=1}^{N_G(e)-1} \alpha'(G, \varnothing, j) f(j)(\mathbb{E}[\theta|1] - \eta') > \sum_{j=1}^{N_G(e)-1} \alpha''(G, \varnothing, j) f(j)(\mathbb{E}[\theta|1] - \eta')$, where α' and α'' corresponds the sender' strategy in the best equilibrium for the DM when $\eta = \eta'$ and $\eta = \eta''$, respectively.

From the fact that α' is supported as an equilibrium when $\eta = \eta'$ implies that (8) holds for all $t \leq N_G(e')$. Since $\eta' > \eta''$, this implies that

$$\widehat{\alpha}\left(G^{N_G(e')-1}, N_G(e') - 1\right) f(N_G(e') - 1)\,\mathbb{E}[\theta|N_G(e') - 1] = \eta'' \sum_{n\geq N_G(e')}^{N} f(n)$$

for some $\widehat{\alpha}\left(G^{N_G(e')-1}, N_G(e')-1\right) < \alpha'\left(G^{N_G(e')-1}, N_G(e')-1\right)$, where we used the notation $\alpha(m^\tau, j) = \Pi_{s=1}^{\tau}\alpha(m_s, m^{s-1}, j)$. Then (8) holds for $t = N_G(e')-1$ implies that

$$\begin{aligned}
&\sum_{j\geq N_G(e')-2}^{N_G(e')-1}\{1-\widehat{\alpha}\left(G^{N_G(e')-1}, j\right)\}\widehat{\alpha}\left(G^{N_G(e')-2}, j\right) f(j)\,\mathbb{E}[\theta|j] \\
= \quad & \eta''[\widehat{\alpha}\left(G^{N_G(e')-1}, N_G(e')-1\right) f(N_G(e')-1) + \sum_{n\geq N_G(e')}^{N} f(n)],
\end{aligned}$$

for some $\widehat{\alpha}\left(G^{N_G(e')-2}, N_G(e')-2\right) \leq \alpha'\left(G^{N_G(e')-2}, N_G(e')-2\right)$ and $\widehat{\alpha}\left(G^{N_G(e')-2}, N_G(e')-1\right) \leq \alpha'\left(G^{N_G(e')-2}, N_G(e')-1\right)$ with at least one strictly inequality. Continuing this, we will eventually get $\widehat{\alpha}(G, j) \leq \alpha'(G, j)$ for all j with at least one strict inequality. This implies that a sender's strategy $\widehat{\alpha}$ can be supported as an equilibrium when $\eta = \eta''$. However, it contradicts α'' being the best equilibrium for the DM since

$$\begin{aligned}
&\sum_{j=1}^{N_G(e'')-1}\widehat{\alpha}(G, \varnothing, j) f(j)(\mathbb{E}[\theta|1]-\eta'') - \sum_{j=N_G(e'')}^{N} f(j)(\mathbb{E}[\theta|1]-\eta'') \\
> \quad & \sum_{j=1}^{N_G(e'')-1}\alpha'(G, \varnothing, j) f(j)(\mathbb{E}[\theta|1]-\eta'') - \sum_{j=N_G(e'')}^{N} f(j)(\mathbb{E}[\theta|1]-\eta'') \\
> \quad & \sum_{j=1}^{N_G(e'')-1}\alpha''(G, \varnothing, j) f(j)(\mathbb{E}[\theta|1]-\eta'') - \sum_{j=N_G(e'')}^{N} f(j)(\mathbb{E}[\theta|1]-\eta'').
\end{aligned}$$

Q.E.D.

8.6 Proof for Section 8

Proof of Theorem 9: A probabilistic commitment is characterized by a $\xi-1$ dimensional vector $\sigma = (\sigma_1, \sigma_2, .., \sigma_{\xi-1})$, where σ_j is the probability that the DM accepts the proposal after requiring j pieces of good evidence. We will prove that for any σ the the probabilistic commitment given in the theorem attains higher expected payoff for the DM. Towards this end, pick a commitment σ and fix it. Also, denote by $\pi(\sigma)$ be the DM's expected payoff associated with commitment σ, and $k(\sigma)$ be the threshold type of sender above which he is eventually accepted by the DM. It is without loss of generality to assume the followings:

$$(1-\sigma_l)\sum_{j\geq l+1}^{N} f(j)\,\eta < \sum_{j\geq l}^{k(\sigma)-1} f(j)\,\mathbb{E}[\theta|j] \text{ for all } l \leq k(\sigma)-1, \tag{33}$$

because otherwise, another commitment $\sigma' = (\sigma_1, \sigma_2, ..\sigma_{l-1}, 1, 1, 1)$ attains higher expected payoff for the DM.

First suppose that $\sigma_1 > \delta/V$. Then, every sender $j \geq 1$ communicates a piece of good evidence at period 1. Then we have $\pi(\sigma) =$

$$\sigma_1\{\sum_{j\geq 1}^{N} f(j)(\mathbb{E}[\theta|j]-\eta)-\eta_S f(0)\}+(1-\sigma_1)\sigma_2\Psi_2+(1-\sigma_1)(1-\sigma_2)\sigma_3\Psi_3$$

$$+\cdot+(1-\sigma_1)(1-\sigma_2)\cdot\left(1-\sigma_{k(\sigma)-1}\right)\{\sum_{j\geq k(\sigma)}^{N} f(j)(\mathbb{E}[\theta|j]-(k(\sigma)-1)\eta)\},$$

where Ψ_j is the expected payoff for the DM when she accepts at period j.

Think of the commitment $\sigma' = (\eta/V, \sigma_2, .., \sigma_\xi)$. Then we have $\pi(\sigma) \geq$

$$\delta/V\{\sum_{j\geq k(\sigma)}^{N} f(j)(\mathbb{E}[\theta|j]-\eta)-\eta_S f(0)\}+(1-\delta/V)\sigma_2\Psi_2$$

$$+\cdot+(1-\delta/V)(1-\sigma_2)\cdot\left(1-\sigma_{k(\sigma)-1}\right)\{\sum_{j\geq k(\sigma)}^{N} f(j)(\mathbb{E}[\theta|j]-(k(\sigma)-1)\eta)-\eta_S\sum_{j<k(\sigma)}^{N} f(j)\},$$

which is strictly higher than $\pi(\sigma)$, because of (33). This implies that for all commitment σ such that $\sigma_1 > \delta/V$, there is a commitment σ' such that $\sigma'_1 = \delta/V$ and attains higher expected payoff for the DM. Applying the same reasoning inductively, we can prove that for all commitment σ such that $\sigma_j > \delta/V$ for some j, there is a commitment σ' such that $\sigma'_j = \delta/V$ for all j and attains higher expected payoff for the DM.

Next, suppose that $\sigma_{k(\sigma)-1} < \delta/V$. Then we have

$$\begin{aligned}\pi(\sigma) &= \sigma_1\Psi_1+(1-\sigma_1)\sigma_2\Psi_2+\cdot+ \\ & \quad (1-\sigma_1)\cdot\cdot\left(1-\sigma_{k(\sigma)-2}\right)\sigma_{k(\sigma)-1}\{\sum_{j\geq k(\sigma)}^{N} f(j)(\mathbb{E}[\theta|j]-(k(\sigma)-1)\eta)\} \\ & \quad (1-\sigma_1)\cdot\cdot\left(1-\sigma_{k(\sigma)-1}\right)\{\sum_{j\geq k(\sigma)}^{N} f(j)(\mathbb{E}[\theta|j]-k(\sigma)\eta)\}.\end{aligned}$$

Think of the commitment $\sigma' = (\sigma_1, ., \sigma_{k(\sigma)-2}, \delta/V, 1, ., \sigma_\xi)$. Then we have

$$\begin{aligned}\pi(\sigma') &= \sigma_1\Psi_1+(1-\sigma_1)\sigma_2\Psi_2+\cdot+ \\ &= (1-\sigma_1)\cdot\cdot\left(1-\sigma_{k(\sigma)-2}\right)\frac{\delta}{V}\{\sum_{j\geq k(\sigma)}^{N} f(j)(\mathbb{E}[\theta|j]-(k(\sigma)-1)\eta)\} \\ &= (1-\sigma_1)(1-\sigma_2)\cdot\left(1-\frac{\delta}{V}\right)\{\sum_{j\geq k(\sigma)}^{N} f(j)(\mathbb{E}[\theta|j]-k(\sigma)\eta)\},\end{aligned}$$

which is strictly higher than $\pi(\sigma)$. Applying the same reasoning inductively backward, we can prove that for all commitment σ such that $\sigma_j < \delta/V$ for some j, there is a commitment σ' such that $\sigma'_j = \delta/V$ for all j and attains higher expected payoff for the DM.

Proof of Theorem 10: We denote the solution to the commitment problem by $r(\eta, \eta_S, \delta)$, and let $\kappa(\eta, \eta_S, \delta)$ be the longest time of talk in the best equilibrium for DM, i.e.,

$$\beta(A, G^{\kappa}) = 1 \text{ and } \beta\left(A, G^{t}\right) = \delta/V \text{ for all } t \leq \kappa - 1.$$

Since the result is trivially true when $\kappa(\eta, \eta_S, \delta) = 0$, think of the case in which $\kappa(\eta, \eta_S, \delta) \geq 1$, i.e., $V^*_{DM}(\varnothing) = \sum_{n=1}^{N} f(n)\,\alpha(G, n, \varnothing)(\mathbb{E}[\theta|n] - \eta) - \eta_S \sum_{n=1}^{N} f(n)(1 - \alpha(G, n, \varnothing))$. Let $r = r(\eta, \eta_S, \delta)$ and $\kappa = \kappa(\eta, \eta_S, \delta)$.

Because $\alpha(G, \varnothing, j) = 1$ for all $j \geq \kappa$, we have

$$\begin{aligned}
& \Upsilon_{DM}(\kappa) - V^*_{DM}(\varnothing) \\
= & \sum_{j \geq \kappa} f(j)(\mathbb{E}[\theta|j] - \kappa\eta) - \eta_S \sum_{j<\kappa} f(j) \\
& - \sum_{j=1}^{N} f(j)\,\alpha(G, \varnothing, j)(\mathbb{E}[\theta|j] - \eta) + \eta_S \sum_{j=1}^{N} f(j)\{1 - \alpha(G, \varnothing, j)\} \\
= & -\sum_{j \geq \kappa} f(j)\,\kappa\eta + \sum_{j=1}^{N} f(j)\,\alpha(G, \varnothing, j)\,\eta \\
& - \sum_{j \geq 1}^{\kappa-1} f(n)\,\alpha(G, \varnothing, j)\,\mathbb{E}[\theta|n] - \eta_S \sum_{j=1}^{k-1} f(j)\,\alpha(G, \varnothing, j),
\end{aligned}$$

where we used $\alpha(G, \varnothing, j) = 1$ for all $j \geq \kappa$. Then it follows that

$$\begin{aligned}
& \Upsilon_{DM}(\kappa) - V_{DM}(\varnothing) \\
= & -\kappa\eta \sum_{j \geq \kappa} f(j) + \eta \sum_{j=1}^{N} f(j)\,\alpha(G, j, \varnothing) - \eta_S \sum_{j=1}^{k-1} f(j)\,\alpha(G, \varnothing, j). \\
& - \sum_{j \geq 1}^{\kappa-1} f(n)\,\alpha(S, j, G)\,\alpha(G, j, \varnothing)\,\mathbb{E}[\theta|j] - \sum_{j \geq 1}^{\kappa-1} f(n)\,\alpha(G, j, G)\,\alpha(G, j, \varnothing)\,\mathbb{E}[\theta|j] \\
= & -\eta\kappa \sum_{j \geq \kappa} f(j) + \eta \sum_{j=1}^{N} f(j)\,\alpha(G, \varnothing, j) - \eta_S \sum_{j=1}^{k-1} f(j)\,\alpha(G, \varnothing, j) \\
& + \eta \sum_{j \geq 1}^{N} f(j)\,\alpha(G, G, j)\,\alpha(G, \varnothing, j) + \eta_S \sum_{j \geq 1}^{\kappa-1} f(j)\,\alpha(S, j, G)\,\alpha(G, j, \varnothing) \\
& - \sum_{j \geq 1}^{\kappa-1} f(j)\,\alpha(G, G, j)\,\alpha(G, \varnothing, j)\,\mathbb{E}[\theta|j] \\
= & \cdot\cdot = -\eta\kappa \sum_{j \geq \kappa} f(j) + \eta \sum_{j \geq 1}^{N} \sum_{t \geq 1}^{\kappa} \Pi_{s=0}^{t-1} \alpha(G, G^s, j)\, f(j) \\
= & \eta \sum_{j \geq 1}^{\kappa-1} \sum_{t \geq 1}^{\kappa} \Pi_{s=0}^{t-1} \alpha(G, G^s, j)\, f(j) \geq 0,
\end{aligned}$$

where the last inequality is strict when $k \geq 1$. Note that we used the conditions of benchmark equilibrium repeatedly, i.e.,

$$-\sum_{j\geq 1} \alpha\left(S, G^{n-1}, j\right) \Pi_{j=0}^{n-1} \alpha\left(G, G^{j}, j\right) \left(\mathbb{E}[\theta|j] + \eta_S\right) = \eta \sum_{j\geq 1}^{\kappa-1} \alpha\left(G, G^{n-1}, j\right) \Pi_{j=0}^{n-1} \alpha\left(G, G^{j}, j\right),$$

for all $n < \kappa$. *Q.E.D.*

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