

On the Efficiency of Stable Matchings in Large Markets

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ABSTRACT. Stability is often the goal for matching clearinghouses, such as those matching residents to hospitals, students to schools, etc. We study the wedge between stability and utilitarian efficiency in large one-to-one matching markets. We show stable matchings are efficient asymptotically for a rich preference class. The speed at which efficiency of stable matchings converges to its optimum depends on the underlying preferences. Furthermore, for severely imbalanced markets governed by idiosyncratic preferences, or when preferences are sub-modular, stable outcomes may be inefficient asymptotically. Our results can guide market designers who care about efficiency as to when standard stable mechanisms are desirable.

Keywords: Matching, Stability, Efficiency, Market Design.

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1. INTRODUCTION

1.1. Overview. Most centralized matching mechanisms do not allow for transfers between participants as an institutional constraint: the National Resident Matching Program (NRMP), clearinghouses for matching schools and students in New York City and Boston, and many others utilize algorithms that ban transfers and implement a stable matching corresponding to reported rank preferences. In fact, in some cases, such as organ donations or child adoption, transfers are viewed not only as “repugnant,” they are banned by law (see Roth, 2007). Even without transfers, stable matchings are appealing in many respects—it is simple to identify one of them once preferences are reported, and they are all Pareto efficient. Furthermore, some work suggests that clearinghouses that implement such stable matchings tend to be relatively persistent (see Roth, 2002; Roth and Xing, 1994).

Nevertheless, the NRMP, for instance, has been subject to complaints from residents regarding the ordinal nature of the mechanism underlying the matching process. These complaints culminated in an official lawsuit filed by a group of resident physicians on May of 2002. The lawsuit alleged that several major medical associations such as the NRMP and the American Council for Graduate Medical Education, as well as numerous prominent hospitals and universities, violated the Sherman antitrust act by limiting competition in the “recruitment, hiring, employment, and compensation of resident physicians” and by imposing “a scheme of restraints which have the purpose and effect of fixing, artificially depressing, standardizing, and stabilizing resident physician compensation and other terms of employment.” The lawsuit effectively highlights the restricted ability of the NRMP to account for marginal (cardinal) preferences of participants over matches (see Crall, 2004).¹ It inspired a flurry of work studying the potential effects the NRMP imposes on wage patterns, as well as on possible modifications to the NRMP that could potentially alleviate the highlighted issues (see Bulow and Levin, 2006, Crawford, 2008, and follow-up literature).

This raises several natural questions: would the availability of transfers affect dramatically the outcomes of stable mechanisms such as the NRMP? Would banning transfers yield stable matchings that are far from efficient? The current paper addresses these questions in the context of large markets. In particular, we characterize a wide class of environments in which

¹Details of the case can be found at http://www.gpo.gov/fdsys/pkg/USCOURTS-dcd-1_02-cv-00873

stable matchings without transfers are asymptotically efficient. In such settings the use of stable mechanisms based on ordinal reports are justified on efficiency grounds and produce outcomes that may be rather close to those produced by stable mechanisms allowing for transfers. We also identify environments in which the tension between efficiency and stability (absent transfers) may be more pronounced.

Our results relate more generally to the question of proper objective functions in the design of matching markets. The literature thus far focused predominantly on mechanisms in which only ordinal preferences are specified. In many other economic settings in which market design has been utilized, such as auctions, voting, etc., utilities are specified for market participants and serve as the primitive for the design of mechanisms maximizing efficiency. For certain matching contexts, such as those pertaining to labor markets, school choice, or real estate, to mention a few, it would appear equally reasonable to assume cardinal assessments.² A designer may then face a trade-off between efficiency and stability, and the availability of transfers providing one way by which to overcome it. Our analysis suggests the types of environments in which this trade-off is not substantial.

In general, when preferences can be represented in utility terms, stable matchings need not be efficient. Indeed, consider a market with two firms $\{f_1, f_2\}$ and two workers $\{w_1, w_2\}$, in which any match between a firm f_i and a worker w_j generates an identical payoff to both (say, as a consequence of splitting the resulting revenue), and all participants prefer to be matched to anyone in the market over being unmatched. Payoffs are given as follows:

	w_1	w_2
f_1	5	4
f_2	3	1

where the entry corresponding to f_i and w_j is the payoff for that pair if matched. Suppose utilities are quasi-linear in money (or transfers). In this case, the unique stable matching matches f_i with w_i , $i = 1, 2$ and generates a utilitarian efficiency of $2 \times (5 + 1) = 12$. However, the alternative matching, between f_i and $w_j, i \neq j, i = 1, 2$ generates a greater utilitarian

²In fact, there is a volume of work that studies matching scenarios in which agents' preferences are cardinal. E.g., in the context of the marriage market, Becker 1973, 1974 and Hitch Hortacsu, and Ariely, 2010; in the context of decentralized matching, Lauermaun, 2013 and Niederle and Yariv, 2009; in the context of assignment problems, Budish and Cantillon, 2012 and Che and Tercieux, 2013; etc.

efficiency of $2 \times (3 + 4) = 14$, and would be the unique stable outcome were transfers available.

When transfers are available, stability is tantamount to utilitarian efficiency (see Roth and Sotomayor, 1992). In all that follows we will refer to stable matchings without transfers as simply stable. The goal of the current paper is then to analyze the wedge between stability and utilitarian efficiency in large markets with sufficient randomness of utilities. Certainly, if we just replicated the 2×2 market above, we could easily generate an arbitrarily large market in which stable matchings lead to a significantly lower utilitarian efficiency than the first best, and transfers could prove useful. The underlying message of the paper is that for a rather general class of preference distributions, such examples of substantial inefficiency are increasingly unlikely as market size grows. Our results therefore suggest when designing mechanisms that exploit the array of benefits stability entails do not come at a great utilitarian cost when markets are sufficiently large. However, the speed at which expected utilitarian efficiency of stable matchings converges to its maximum varies for different preference distributions. Aligned and assortative preferences lead to fairly fast convergence, while idiosyncratic preferences are associated with somewhat slower convergence.

In order to glean some intuition for our results, consider the case in which there are n firms and n workers, all preferring to be matched to someone in the market over remaining unmatched, and that the utility of any firm f_i and worker w_j coincides for both and is randomly and independently determined according to a uniform distribution between 0 and 1. In other words, a market is specified by the realization of n^2 uniform variables corresponding to every firm and worker pair.³

Now, for any $\varepsilon > 0$, construct the following bipartite graph, where firms correspond to one type of nodes, and workers correspond to the other type of nodes. Any worker and firm are linked if the utility their matching generates is greater than $1 - \varepsilon$. For instance, Figure 1 corresponds to a 4×4 market, where match utilities are described in the left panel, and the induced bipartite graph for $\varepsilon = 0.25$ is depicted on the right with bold lines (bold numbers in the left panel correspond to pairs with utilities greater than $1 - \varepsilon$).

The probability of a link between any firm f_i and worker w_j is given by ε . Therefore, an old result from graph theory due to Erdos and Renyi (1964) suggests that, as n becomes

³Throughout, we assume agents have quasi-linear utilities and abuse labeling by referring to the match utilities absent transfers simply as “utilities.”

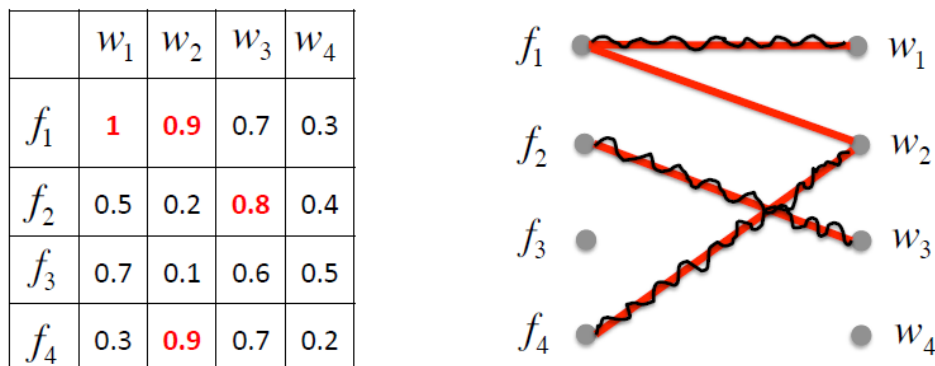


Figure 1: Heuristic Construction of Almost Efficient and Stable Matchings

arbitrarily large, if ε is decreasing with n at a rate slower than $\frac{\log n}{n}$, so that the induced networks are sufficiently connected, they will contain a *perfect matching*, one corresponding to a one-to-one matching connecting all firms and workers, with probability converging to 1 (in Figure 1, the squiggly connections correspond to a matching that is almost perfect).

In such a perfect matching, the maximal benefit any agent can gain by connecting with an alternative participant is at most ε and so the matching is “almost stable.” In addition, the per-person utility is greater than $1 - \varepsilon$, so that the matching is also “almost efficient.”⁴

In fact, we show that, *the (generically unique) fully stable matching is asymptotically efficient*. Furthermore, we illustrate that the convergence speed is given by $\frac{\log n}{n}$.

Identical conclusions to those emerging in the full alignment case follow for settings in which firms and workers split their match surpluses using a fixed sharing rule. Such settings are common in many applications (see, e.g., Sorensen, 2007, and the literature that followed). Nonetheless, there are many settings in which individuals have idiosyncratic preferences over partners. For instance, employees may have idiosyncratic preferences over locations of their employers, while employers may have idiosyncratic preferences over the particular portfolios of potential employees. Propositions 1 and 2 provide general results for settings in which the utilities each pair of participants experiences is a function of both a common shock, as

⁴An alternative proof establishing this observation can be deduced from the appendix of Compte and Jehiel (2008).

well as individual idiosyncratic shocks, in a fairly arbitrary manner (in particular, shocks are not necessarily uniformly random). We show that *all* stable matchings are asymptotically efficient in these settings. Nonetheless, idiosyncrasies reduce the speed of convergence. When utilities are fully idiosyncratic and uniform, the difference between efficiency generated by stable matchings and the maximal feasible efficiency is of the order of $\frac{1}{\log n}$.

The last case we consider is that in which each side of the market may share preferences over members of the other side of the market. For example, medical residents may share preferences over hospitals that are due to their published rankings and hospitals may agree on what makes a medical resident desirable.⁵ In purely assortative markets as such, all matchings are utilitarian efficient. Our results here illustrate that in markets with preferences that are hybrids of assortative and idiosyncratic, stable matchings are asymptotically efficient. Furthermore, when agents are characterized by preferences that are non-trivial convex combinations of assortative and idiosyncratic components, convergence to efficiency still holds and the speed of convergence is of the order of $n^{-1/4}$, substantially higher than that corresponding to preferences that are convex combinations of aligned and idiosyncratic components.

In all such environments, a designer using stability as an objective would be justified on efficiency grounds as well, at least when markets are large enough. In particular, a ban on transfers would not come at a great efficiency cost. Naturally, the speed of convergence plays an important role in determining which markets are likely to be “large enough.” Our results suggest that market size requirements would be less stringent for markets that have dominant aligned or assortative preference components.⁶

There are a few further caveats to the message conveyed by our first main results on the asymptotic efficiency of stable matchings. The first caveat has to do with the notion of efficiency one uses. The notion we considered thus far is that of expected efficiency per market participant. This notion gives a normalized assessment of the expected outcomes participants

⁵Agarwal (2015) reports that conversations with residency program and medical school administrators indicate that, indeed, programs broadly agree on what makes a resident appealing.

⁶The NRMP is a leading example of a large one-to-one matching market. Results from the NRMP are in line with our insights here. In the 2016 installment of the NRMP, looking at matched US seniors, 53% of applicants were matched with their first-ranked hospital and 85.1% were matched with one of their four top-ranked hospitals, with similar figures appearing for independent applicants. Furthermore, there is a negative association between the number of applicants in a specialty, reflecting the size of the relevant sub-market, and the average rank of matched programs (where lower ranks correspond to more preferred programs).

are likely to experience in stable mechanisms. In particular, it reflects individual incentives to switch from one potential clearinghouse to another (e.g., a switch from clearinghouses allowing transfers to ones that ban them). Nonetheless, market designers may also be concerned about overall efficiency, reflecting, say, the overall costs of banning transfers in clearinghouses such as the NRMP. We therefore also analyze the unnormalized asymptotic efficiency of stable matchings. The benchmark maximum efficiency achievable through any matching is a solution of a variation of the *optimal assignment problem* in statistics (see, e.g., Walkup, 1979 and work that followed). We show that for the classes of preferences we study, when considering unnormalized efficiency, stable matchings are substantially less efficient than the optimum for any market size. Furthermore, idiosyncratic preference components seem to generate greater wedges between the most efficient matchings and the stable ones.

Second, there are certain market features that may induce stable matchings whose efficiency is bounded away from the optimum even when markets are large and one considers normalized efficiency. One such feature is market imbalances. Indeed, many real-world markets contain unequal volumes of participants on both sides of the market. From a theoretical perspective, recent work suggests that an imbalance in the market gives a disadvantage to the abundant side of the market (see Ashlagi, Kanoria, and Leshno, 2017). In our setting, a bounded difference between the volumes on either side does not change our conclusions that stable matchings are asymptotically efficient. These results also continue to hold for unbounded volume differences when preferences are aligned or assortative. However, in Section 5.1 we show that whenever utilities are determined in a fully random fashion and the difference between the volumes on the two sides of the market increases at least linearly in the size of the market, stable matchings may not be asymptotically efficient. The other feature of markets that may make stable matchings be inefficient even asymptotically has to do with preferences. While the classes of preferences we focus on in the paper (namely, hybrids of aligned, assortative, and idiosyncratic components) are some of the most prominent in the literature, our results do not hold globally. For instance, Becker (1974) already pointed out the impact of preference modularity on the utilitarian efficiency of stable matchings. In Section 5.2 we show that in assortative markets with sub-modular preferences, stable matchings may

entail a substantial amount of inefficiency regardless of the size of the market.⁷

To summarize, taken together, our results can provide guidance to market designers who care about efficiency, or contemplate the introduction of some form of transfers between participants, as to when standard stable mechanisms are desirable. If a designer’s concern regards expected outcomes per participant, and markets are fairly balanced, stable mechanisms are justified on efficiency grounds for sufficiently large markets. In particular, the availability of transfers will not affect outcomes significantly. However, if markets are not very large, or severely imbalanced and entailing a prominent idiosyncratic component of participants’ preferences, or if the designer worries about overall (unnormalized) outcomes, commonly used stable mechanisms may not be ideal.

1.2. Literature Review. There are several strands of literature related to this paper. Efficiency of stable matching has been a topic of recent study. Boudreu and Knoblauch (2010) provide an upper bound on the sum of partner ranks in stable matchings when preferences exhibit particular forms of correlation. Consistent with our results, these upper bounds increase at a speed slower than the size of the market.⁸

Several papers have considered the utilitarian welfare loss stability may entail in matching markets. Anshelevich, Das, and Naamad (2009) consider finite markets and particular constellations of utilities. They provide bounds on the utilitarian efficiency achieved through stability relative to that achieved by the efficiency-maximizing matching. Compte and Jehiel (2008) consider a modified notion of stability taking into account a default matching and suggest a mechanism that produces an “optimal” such matching that is asymptotically efficient when preferences are fully idiosyncratic and drawn from the uniform distribution (in line with our Proposition 1). Durlauf and Seshadri (2003) consider assortative markets in which agents may form coalitions, of any size, whose output depends on individuals’ ability profile. Their

⁷In these assortative markets, each individual is characterized by an ability, and a pair’s utility coincides with their “output,” which increases in both of their abilities. Sub-modularity then means that the marginal increase in output with respect to a match’s ability is decreasing in one’s own ability.

⁸In a related paper, Knoblauch (2007) illustrates bounds on expected ranks for participants when one side of the market has uniformly random preferences and the other has arbitrary preferences. Liu and Pycia (2016) consider ordinally efficient mechanism and illustrate that uniform randomizations over deterministic efficient mechanisms in which no small group of agents can substantially change the allocation of others are asymptotically ordinally efficient, thereby showing that ordinal efficiency and ex-post Pareto efficiency become equivalent in large markets, and that many standard mechanisms are asymptotically ordinally efficient.

results imply that the (utilitarian) efficiency of assortative matchings depends on the presence of positive cross-partial derivatives between the abilities of the partners in the output of a marriage, in line with our results (see Section 5.2).⁹

Che and Tercieux (2015) study assignment problems in which individual agents have utilities that are composed of a valuation common to all agents and idiosyncratic individual shocks (analogous to our hybrid model of assortative and idiosyncratic preferences, studied in Section 4). They show that Pareto efficient allocations are asymptotically utilitarian efficient. However, in the case of assignment problems they study, stable allocations are not necessarily Pareto efficient, so they are not necessarily utilitarian efficient. In contrast, we study general one-to-one matching markets (allowing, in particular, for aligned and/or idiosyncratic preferences on both sides, as well as rather general imbalanced markets). In our setting of two-sided matching, stable matchings are utilitarian efficient.

Our results focus on large markets, which have received some attention in the literature, mostly due to the observation that many real-world matching markets involve many participants (e.g., the NRMP that involves several tens of thousands of participants each year, schooling systems in large cities, etc.). The literature thus far has mostly focused on incentive compatibility constraints imposed by stable matching mechanisms when markets are large; see for instance Immorlica and Mahdian (2005), Kojima and Pathak (2009), and Lee (2017). While most of this literature focuses on balanced markets, recently Ashlagi, Kanoria, and Leshno (2017) have noted that imbalances in the volume of participants on both sides of the market may alleviate incentive compatibility issues, particularly when markets are large. We use some of their results when we discuss the efficiency in imbalanced markets in Section 5.1.

Our paper also relates to the “price of anarchy” notion introduced in Computer Science (see Roughgarden and Tardos, 2007). In general, the price of anarchy is defined as the ratio between the social utility of the (worst) Nash equilibrium outcome of a game and the maximum social utility possible in that game. In our context, a natural substitute to Nash equilibrium

⁹Dizdar and Moldovanu (2016) study a two-sided matching model in which agents are characterized by privately known, multi-dimensional attributes that jointly determine the “match surplus” of each potential partnership. They assume utilities are quasi-linear, and monetary transfers among agents are feasible. Their main result shows that the only robust rules compatible with efficient matching are those that divide realized surplus in fixed proportions, independently of the attributes of the pair’s members. Our results illustrates that, when markets are large, transfers can be banned altogether and outcomes arbitrarily close to the optimum can be achieved.

is a stable matching. In that respect, our results characterize the price of anarchy in many one-to-one matching environments. In particular, when considering normalized efficiency, the asymptotic price of anarchy is 1 for a wide array of balanced markets.¹⁰

There is a large body of literature studying efficiency of mechanisms in other realms, such as auctions (see Chapter 3 in Milgrom, 2004) or voting (see Krishna and Morgan, 2015). The current paper provides an analogous study in the context of one-to-one matching.

Methodologically, our results borrow techniques introduced by Knuth (1976), Walkup (1979), Pittel (1989, 1992), and Lee (2017) and the intuition for our main insights utilizes graph theoretical results first noted by Erdos and Renyi (1964).

2. THE MODEL

Consider a market of n firms $F = \{f_1, \dots, f_n\}$ and n workers $W = \{w_1, \dots, w_n\}$ who are to be matched with one another. At the outset, two $n \times n$ matrices $(u_{ij}^f)_{i,j}$ and $(u_{ij}^w)_{i,j}$ are randomly determined according to a non-atomic probability distribution \mathcal{G} over $[0, 1]^{2n^2}$. When firm f_i and worker w_j match, they receive match utilities u_{ij}^f and u_{ij}^w , respectively. We assume that any agent remaining unmatched receives a match utility of 0, so that all agents prefer, at least weakly, to be matched with any agent over remaining unmatched (and this preference is strict almost always).¹¹ We further assume that utilities are quasi-linear in money.¹² This specification allows for a natural benchmark of utilitarian efficiency that corresponds to any stable matching with transfers. Throughout the paper, we will often abuse labeling and refer to u_{ij}^f and u_{ij}^w as utilities.

We consider market matchings $\mu : F \cup W \rightarrow F \cup W$ such that for any $f_i \in F$, $\mu(f_i) \in W$, for any $w_j \in W$, $\mu(w_j) \in F$, and if $\mu(f_i) = w_j$ then $\mu(w_j) = f_i$. We will at times abuse

¹⁰Without restricting preferences in any way, and taking a worst-case point of view, Echenique and Galichon (2015) show that the price of anarchy can be arbitrarily low (i.e., for any value, one can always find a market in which stability in the non-transferable utility model produces efficiency lower than the optimum by at least that value).

¹¹We relax the assumptions that utilities are drawn from distributions with bounded support and that all agents are acceptable in the Online Appendix.

¹²That is, whenever firm f_i matches with worker w_j and transfers an amount m to the worker, the respective utilities for the firm and worker are given by:

$$U_{f_i}(w_j; m) = u_{ij}^f - m \text{ and } U_{w_j}(f_i; m) = u_{ij}^w + m.$$

notation and denote $\mu(i) = j$ and $\mu(j) = i$ if $\mu(f_i) = w_j$. Denote by M the set of all market matchings. For any realized match utilities u_{ij}^f and u_{ij}^w , *stable matchings* are a subset of M that satisfies the following condition: For any firm and worker pair (f_i, w_j) , either $u_{i\mu(i)}^f \geq u_{ij}^f$ or $u_{\mu(j)j}^w \geq u_{ij}^w$. In other words, at least one of the members of the pair (f_i, w_j) prefer their allocated match under μ over their pair member.¹³ Whenever there exist a firm and a worker that prefer being matched to one another over their allocated match partners, the matching under consideration is unstable and that pair is referred to as a *blocking pair*.

In most applications, centralized clearinghouses are designed to implement stable matchings. Our focus in this paper is therefore in assessing the relative utilitarian efficiency of stable matchings to the maximal utilitarian efficiency achievable in a market matching.

The expected maximal utilitarian efficiency achievable across all market matchings is denoted by E_n :

$$E_n = \mathbb{E}_{\mathcal{G}} \max_{\mu \in M} \sum_{i=1}^n \left(u_{i\mu(i)}^f + u_{i\mu(i)}^w \right).$$

Since stable matchings are not necessarily unique, and utilities of firms and workers are not necessarily symmetric, we denote the worst-case efficiency of stable matchings for firms and workers as follows:

$$S_n^f = \mathbb{E}_{\mathcal{G}} \min_{\{\mu \in M | \mu \text{ is stable}\}} \sum_{i=1}^n u_{i\mu(i)}^f \quad \text{and} \quad S_n^w = \mathbb{E}_{\mathcal{G}} \min_{\{\mu \in M | \mu \text{ is stable}\}} \sum_{i=1}^n u_{i\mu(i)}^w.$$

The minimal utilitarian efficiency achievable by implementing a stable matching for any realized match utilities is denoted by S_n and defined as:

$$S_n = \mathbb{E}_{\mathcal{G}} \min_{\{\mu \in M | \mu \text{ is stable}\}} \sum_{i=1}^n \left(u_{i\mu(i)}^f + u_{i\mu(i)}^w \right) \geq S_n^f + S_n^w.$$

The maximal feasible efficiency per participant is $\frac{E_n}{2n}$. The worst-case efficiency per participant generated by stable matchings is $\frac{S_n}{2n}$. Our goal is to characterize settings in which these two values become close when markets are very large, namely settings such that $\lim_{n \rightarrow \infty} \frac{E_n - S_n}{2n} = 0$,

¹³In general, stability also entails an individual rationality restriction (so that no agent prefers remaining unmatched over her prescribed match). Given our assumptions on utilities, this restriction is automatically satisfied.

in which case we say that stable matchings are *asymptotically efficient*. Notice that the per-person expected utility when a stable matching is implemented is always bounded by the maximal value of the support of match utilities, $\frac{S_n}{2n} \leq 1$ for all n . In particular, whenever $\frac{S_n}{2n} \rightarrow 1$, stable matchings are asymptotically efficient.

A few notes on our underlying model. First, while we phrase our results with the labeling of “firms” and “workers”, they pertain to pretty much any two-sided one-to-one matching environment in which a centralized clearinghouse could be utilized. Having said that, further analysis would be required to handle many-to-one matching environments such as school choice.¹⁴ Second, for most of the paper we will consider the case of a balanced market (n agents on each side). All of our results go through when there is a bounded gap between the volume of firms and workers (say, there are n firms and $n + k$ workers, where k is fixed) with essentially identical proofs. When the difference in volumes is increasing (say, there are n firms and $n + k(n)$ workers, where $k(n)$ is increasing in n), some subtleties arise that we discuss in Subsection 5.1. For presentation simplicity, we initially focus our analysis on balanced markets. Last, while we assume throughout the paper that utilities are drawn from bounded supports and are such that all participants view all partners as acceptable, our results do not hinge on these assumptions. In the Online Appendix, we relax these restrictions. The appendix to the paper contains all proofs.

3. GENERAL ALIGNED MARKETS WITH IDIOSYNCRATIC SHOCKS

We first focus on markets that entail aligned or common impacts on utilities (say, the revenue a worker and firm can generate together) as well as idiosyncratic ones (say, ones corresponding to the geographical location of an employer, or the precise courses a potential employee took in college). We consider general markets allowing for both components. We illustrate that stable matchings in such markets are asymptotically efficient. We also characterize the speed at which the per-person utilitarian efficiency of stable matchings converges to 1. In particular, we show that the speed of convergence is substantially faster as the idiosyncratic component of preferences vanishes.

¹⁴There are also many details that may be important in such environments that our analysis does not handle, for example pre-match investment in perceived quality in the form of test preparation, geographic relocation, and the like (see, e.g., Cole, Mailath, and Postlewaite, 2001 and Avery and Pathak, 2015).

3.1. Asymptotic Efficiency. Formally, we consider utility realizations such that each pair (f_i, w_j) receives a utility that is a combination of the pair's common surplus c_{ij} and independent utility “shocks” z_{ij}^f and z_{ij}^w . That is,

$$\begin{aligned} u_{ij}^f &\equiv \phi(c_{ij}, z_{ij}^f) \quad \text{and} \\ u_{ij}^w &\equiv \omega(c_{ij}, z_{ij}^w). \end{aligned}$$

We assume the functions $\phi(\cdot, \cdot)$ and $\omega(\cdot, \cdot)$, from \mathbb{R}_+^2 to \mathbb{R}_+ , are continuous. We further assume that they are both either strictly increasing in the common component c_{ij} but independent of the idiosyncratic components, z_{ij}^f and z_{ij}^w ; or strictly increasing in the idiosyncratic components but independent of the common component; or strictly increasing in both arguments. In this way, we capture markets characterized by pure aligned preferences, markets in which preferences are fully idiosyncratic, and markets that are non-trivial hybrids of these two former cases. Each of c_{ij} , z_{ij}^f , and z_{ij}^w is drawn independently from a non-trivial distribution over \mathbb{R}_+ , and the three distributions have positive density functions and bounded supports.

Our main result shows that, asymptotically, market participants achieve, on average, their maximal conceivable match utility, regardless of which stable matchings are selected.

Proposition 1 [Efficiency of Stable Matchings]. *Let \bar{c} , \bar{z}^f , and \bar{z}^w be upper bounds for the support of the distributions of c_{ij} , z_{ij}^f , and z_{ij}^w , respectively. Then,*

$$\lim_{n \rightarrow \infty} \frac{S_n^f}{n} = \phi(\bar{c}, \bar{z}^f) \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{S_n^w}{n} = \omega(\bar{c}, \bar{z}^w).$$

Therefore, $\lim_{n \rightarrow \infty} \frac{S_n}{n} = \phi(\bar{c}, \bar{z}^f) + \omega(\bar{c}, \bar{z}^w)$.

Proposition 1 illustrates that *fully stable matchings are asymptotically efficient, even for utilities that are arbitrary combinations of common and idiosyncratic components that are realized from arbitrary continuous distributions*. An indirect consequence of the proposition is that the most efficient matching asymptotically achieves the maximal conceivable utility per participant.

Some prior work (most notably, Abdulkadiroglu, Pathak, and Roth, 2009 in the context of school choice) has suggested that top trading cycle mechanisms can considerably improve upon

deferred acceptance algorithms, even in large markets, when considering matched partners' ranks. These results are consistent with ours. For illustration, consider the case in which agent's utilities are completely random: each firm's utility from each worker, and each worker's utility from each firm are determined using independently and identically distributed uniform draws from $[0, 1]$. Our results suggest that for any $\varepsilon > 0$, in large enough markets, most agents will be matched with an agent achieving at least $1 - \varepsilon$ in utility terms. So, while agents will not necessarily be matched with their very top candidates, they will match with fairly close substitutes to them.¹⁵ As we show in the Online Appendix, this observation is not an artifact of our assumption that utilities are drawn from bounded supports, though when the support of utilities expands rapidly enough, asymptotic efficiency fails, as we will see in Section 3.4.

We soon describe some intuition for this result. Before we do so, we discuss the speeds with which the limits in Proposition 1 are achieved.

3.2. Speed of Convergence. We now turn to the speed of convergence pertaining to the efficiency of stable matchings. As it turns out, the structure of preferences is crucial. In order to provide a characterization of the convergence speed, we restrict attention to linear functions ϕ and ω . Namely, for each firm f_i and worker w_j ,

$$\begin{aligned}\phi(c_{ij}, z_{ij}^f) &= (1 - \alpha)c_{ij} + \alpha z_{ij}^f \quad \text{and} \\ \omega(c_{ij}, z_{ij}^w) &= (1 - \alpha)c_{ij} + \alpha z_{ij}^w,\end{aligned}$$

where $\alpha \in [0, 1]$. We further assume that c_{ij} , z_{ij}^f , and z_{ij}^w are all uniformly distributed over $[0, 1]$.

Proposition 2. *1. If $\alpha = 0$, then for any $n \geq 3$,*

$$\frac{1}{2} \frac{\log n}{n} \leq 1 - \frac{S_n}{2n} \leq \frac{\log n}{n}.$$

¹⁵In fact, simulations of such markets suggest that even for a market with 1000 participants on each side, top-trading-cycles improve the ranking of matched partners under deferred acceptance for over 50% of market participants, but improve average utilities of participants by about 1% within the $[0, 1]$ range.

2. If $\alpha > 0$,

$$\limsup_{n \rightarrow \infty} \left(1 - \frac{S_n^f}{n}\right) \log n = \limsup_{n \rightarrow \infty} \left(1 - \frac{S_n^w}{n}\right) \log n \leq 2.$$

The proposition suggests that the speed of convergence is sufficiently faster when preferences are aligned. Indeed, when $\alpha = 0$, preferences depend solely on the common component c_{ij} . The proposition implies that the speed of convergence of $\frac{S_n}{2n}$ in this case is of the order of $\frac{\log n}{n}$. In contrast, whenever $\alpha > 0$, there is positive weight on the idiosyncratic component. In this case, there might be multiple stable matchings and we consider $\frac{S_n^f}{n}$ and $\frac{S_n^w}{n}$ separately. Their speed of convergence is of the order of $\frac{1}{\log n}$.¹⁶

Figure 2 illustrates numerical results for the utilitarian efficiency per market participant for different levels of α . For each market size n , we run 100 simulations, each corresponding to one realization of preferences in the market. For each simulation, we compute the average per-participant utility induced by their *least* preferred stable match partners. The solid black line, the long dashed line, and the short dashed line depict, respectively, the mean, the 95'th percentile, and the 5'th percentile of the simulated distribution of these averages across the 100 simulations.¹⁷ The solid red line depicts the mean maximal efficiency feasible across the realized markets. As one might expect, greater values of α are associated with lower speeds of convergence. There are two features to note in the figure. First, even for high levels of α , the fraction of the maximal achievable efficiency that is achieved through stability is substantial in this worst-case scenario. For markets with around 1000 participants on each side—much smaller than many of the markets in the applications we discuss—that fraction is about 88% even when $\alpha = 1$, and much higher for lower α . Second, the relationship between that fraction is not linear in α . For instance, for markets with around 1000 participants on each side, that

¹⁶In fact, when $\alpha = 1$, we show in the Online Appendix that

$$\lim_{n \rightarrow \infty} \left(1 - \frac{S_n^f}{n}\right) \log n = \lim_{n \rightarrow \infty} \left(1 - \frac{S_n^w}{n}\right) \log n = 1.$$

¹⁷In principle, when preferences are perfectly aligned, $\alpha = 0$, we can use the formula developed in the section that follows to calculate the expected per-person expected utility. The figure gives a sense of the spread of the distribution (and the mean tracks closely that generated by the formal expression of expected efficiency).

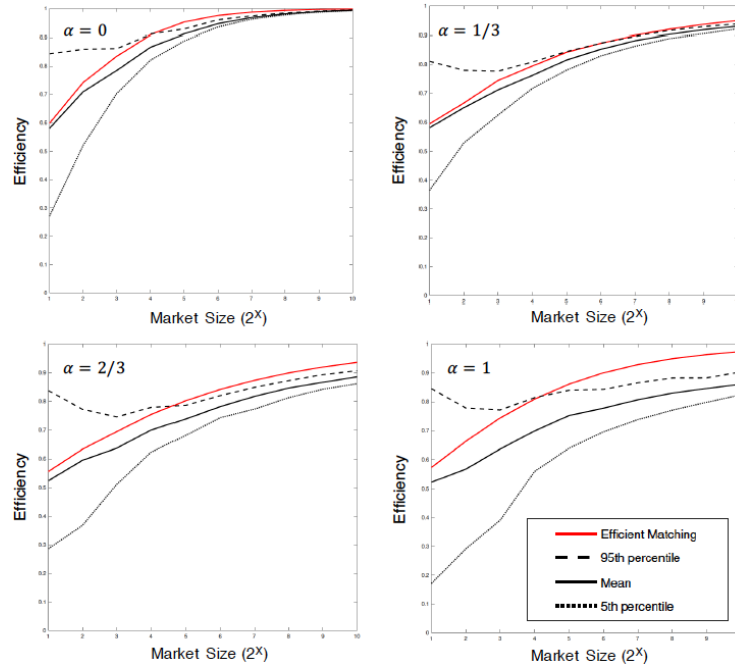


Figure 2: Convergence Speeds for Linear Hybrid Models of Aligned and Idiosyncratic Preferences (α is the Weight on the Idiosyncratic Component)

fraction is about 98% for $\alpha = 1/3$, about 95% for $\alpha = 2/3$, and, as mentioned, about 88% for $\alpha = 1$.

Ultimately, Propositions 1 and 2 combined illustrate that when match utilities have either or both aligned and idiosyncratic components, any selection of stable matchings will lead to matchings that are approximately utilitarian efficient for sufficiently large markets. Nonetheless, the speed of convergence depends heavily on the structure of preferences: as the idiosyncratic component becomes more important, markets need to be larger to achieve approximate efficiency.

3.3. Intuition Underlying Propositions 1 and 2. In this section, we provide a heuristic construction underlying the proofs of Propositions 1 and 2. As it turns out, the case in which preferences are purely aligned requires qualitatively different techniques than the case involving idiosyncratic elements. We thus describe them separately.

Full Alignment. We start with the fully aligned case. In this case, members of each matched pair receive utilities proportional to one another (e.g., a firm and a worker may be

splitting the revenues their interaction generates). As mentioned in the Introduction, such settings are common in many applications (see, e.g., Sorensen, 2007, and the literature that followed).

Formally, we assume here that the utility both firm f_i and worker w_j receive if they are matched is given by $u_{ij} \equiv u_{ij}^f = u_{ij}^w$. We assume u_{ij} are independently drawn across all pairs (i, j) from a continuous distribution G over $[0, 1]$. It follows that, generically, utility realizations $(u_{ij})_{i,j}$ entail a unique stable match. Indeed, consider utility realizations $(u_{ij})_{i,j}$ such that no two entries coincide, and take the firm and worker pair (f_i, w_j) that achieve the maximal match utilities, $\{(i, j)\} = \arg \max_{(i', j')} u_{i'j'}$. They must be matched in any stable matching since they both strictly prefer one another over any other market participant. Consider then the restricted market absent (f_i, w_j) and the induced match utilities on the remaining participants. Again, we can find the pair that achieves the maximal match utility within that restricted market. As before, that pair must be matched in any stable matching. Continuing recursively, we construct the unique stable matching.

The proof of Proposition 1 in this case proceeds as follows. We first consider the uniform distribution. When determining the match utilities, the greatest realized entry, which corresponds to the first matched pair in the construction of the generically unique stable matching mentioned above is the extremal order statistic of n^2 entries. Since each entry is uniform, the expected value of the maximal entry is given by $\frac{n^2}{n^2+1}$. In the next step of our construction, we seek the expected maximal value within the restricted market (derived by extracting the firm and worker pair that generates the highest match utility). That value is the extremal order statistic of $(n-1)^2$ uniform random numbers that are lower than the entry chosen before, and can be shown to have the expected value $\frac{n^2}{n^2+1} \frac{(n-1)^2}{(n-1)^2+1}$. Continuing recursively, it can be shown that:

$$\frac{S_n}{2} = \frac{n^2}{n^2+1} + \frac{n^2}{n^2+1} \frac{(n-1)^2}{(n-1)^2+1} + \frac{n^2}{n^2+1} \frac{(n-1)^2}{(n-1)^2+1} \frac{(n-2)^2}{(n-2)^2+1} + \dots$$

While corresponding summands become smaller and smaller as we proceed with the recursive process above, there are enough summands that are close enough to 1 so that $\lim_{n \rightarrow \infty} \frac{S_n}{2n} = 1$, which is what the proof illustrates.

We then show that our result regarding asymptotic efficiency does not depend on the

uniform distribution of utilities.¹⁸ However, for Proposition 2, in order to show that the utilitarian efficiency of stable matchings converges to 1 at a speed of the order of $\frac{\log n}{n}$, we use the precise formulation of S_n above.

Fully Independent Preferences. The other extreme case of Propositions 1 and 2 has to do with the polar case in which preferences are determined in a fully independent manner. That is, all values $(u_{ij}^f)_{i,j}$ and $(u_{ij}^w)_{i,j}$ are independently and identically distributed according to a continuous distribution G over $[0, 1]$.

Notice first that the maximal conceivable efficiency level (of 1 per person) is achievable asymptotically. Indeed, for any market realization $(u_{ij}^f, u_{ij}^w)_{i,j}$ consider the matrix of utility realizations $\tilde{u}_{ij} \equiv \frac{u_{ij}^f + u_{ij}^w}{2}$. The distribution G then induces a continuous distribution $\tilde{G}[0, 1]$ from which \tilde{u}_{ij} are independently drawn. We can now view $(\tilde{u}_{ij})_{i,j}$ as match utilities shared by firms and workers as in our analysis in Section 3.3. Asymptotic efficiency of stable matchings in that case implies that the unique stable matching of the induced fully aligned market is efficient asymptotically. Note that any matching μ generates the same levels of efficiency when each firm f_i and worker w_j both receive \tilde{u}_{ij} when matching to one another, or when the firm receives u_{ij}^f and the worker receives u_{ij}^w . Therefore,

$$\lim_{n \rightarrow \infty} \frac{E_n}{2n} = 1.$$

In this setting, stable matchings are not generically unique. In order to first glean some intuition for why idiosyncratic individual shocks to preferences may still be consistent with asymptotic efficiency, consider a similar construction to that described in the Introduction for the case of full alignment. For simplicity, consider the case in which G is uniform over $[0, 1]$. We look at the bipartite graph in which firms and workers constitute the two types of nodes. Now, for any $\varepsilon > 0$, a link between firm f_i and worker w_j is formed if *both* $u_{ij}^f > 1 - \varepsilon$ and $u_{ij}^w > 1 - \varepsilon$. In particular, a link is formed with probability ε^2 . Using Erdos and Renyi (1964), there is an asymptotic perfect matching in the induced graph as long as ε^2 converges to 0

¹⁸This effectively relies on the speedy convergence of extremal order statistics for the distributions we consider here. In fact, in the Online Appendix, we show that asymptotic efficiency holds even when considering a class of utility distributions that are not bounded and allowing for agents to view certain partners as unacceptable.

sufficiently slowly so that the graph is ‘connected enough.’ Namely, as long as ε^2 converges to 0 slower than $\frac{\log n}{n}$, or ε converges to 0 slower than $\sqrt{\frac{\log n}{n}}$, the probability of the induced graph containing a perfect matching converges to 1. Notice that in such a matching almost all agents can gain at most ε from pairing with someone other than their allocated match partner. In other words, this is a way to construct an almost stable and almost efficient matching. Nonetheless, an ‘almost stable’ matching may be very far, in terms of number of blocking pairs, overlap in matched pairs, etc. from any of the market’s stable matchings. As mentioned in the introduction, this distinction could be particularly important given that most centralized clearinghouses utilized in matching markets are designed to implement a fully stable matching (assuming participants report truthfully their preferences).

The formal proof of Proposition 1 for this case utilizes different techniques than those employed to prove the proposition for the fully aligned case. It relies on results by Pittel (1989). In order to see the basis of the proof, suppose utilities are drawn from the uniform distribution. Notice that from the ex-ante symmetry of the market, each firm f_i (respectively, each worker w_j) has equal likelihood to be ranked at any position in any worker’s (respectively, firm’s) preference list. Therefore, each one of $n!$ matches of n firms and n workers has the same probability P_n of being stable. Knuth (1976) proved that

$$P_n = \underbrace{\int_0^1 \cdots \int_0^1}_{2n} \prod_{1 \leq i \neq j \leq n} \left(1 - (1 - u_{ii}^f)(1 - u_{jj}^w) \right) d\mathbf{u}_{ii}^f d\mathbf{u}_{jj}^w$$

where $d\mathbf{u}_{ii}^f = du_{11}^f du_{22}^f \cdots du_{nn}^f$ and $d\mathbf{u}_{jj}^w = du_{11}^w du_{22}^w \cdots du_{nn}^w$.

The intuition behind this formula is simple. The formula essentially evaluates the probability that the match μ , with $\mu(i) = i$ for all i , is stable. For any realized market, in order for μ to be stable, utilities $(u_{ij}^f, u_{ij}^w)_{1 \leq i \neq j \leq n}$ must satisfy that either $u_{ij}^f \leq u_{ii}^f$ or $u_{ij}^w \leq u_{jj}^w$ for all $i \neq j$. The integrand corresponds to the probability that these restrictions hold.

Take any $\varepsilon > 0$. Let $P_{\varepsilon, n}$ be the probability that μ is stable and the sum of firms’ utilities is less than or equal to $(1 - \varepsilon)n$. That is,

$$P_{\varepsilon, n} = \int_{\substack{\mathbf{0} \leq \mathbf{u}_{ii}^f, \mathbf{u}_{jj}^w \leq \mathbf{1} \\ \sum_{i=1}^n u_{ii}^f \leq (1-\varepsilon)n}} \prod_{1 \leq i \neq j \leq n} \left(1 - (1 - u_{ii}^f)(1 - u_{jj}^w) \right) d(\mathbf{u}_{ii}^f, \mathbf{u}_{jj}^w). \quad (1)$$

From symmetry, the probability that any matching is stable and the sum of firms' utilities is at most $(1 - \varepsilon)n$ coincides with $P_{\varepsilon,n}$. Since there are $n!$ possible matchings, it suffices to show that $n!P_{\varepsilon,n}$ converges to 0 as n increases. Our proof then uses the techniques developed in Pittel (1989) to illustrate this convergence.¹⁹ When utilities are distributed uniformly, we further show in the Appendix that the convergence speed of $1 - \frac{S_n^f}{n}$ is of the order of $\frac{1}{\log n}$.

We note that Proposition 1 for the hybrid model is not a direct generalization of the arguments used for the two polar cases above. In order to get a sense of the difficulty introduced by combining aligned preferences with idiosyncratic shocks, consider equation 1 above. Roughly speaking, alignment introduces a positive correlation between match utilities (in fact, the relevant match utilities in equation 1 are *positively associated*, see Esary, Proschan, and Walkup, 1967). This positive correlation affects both the integrand as well as the conditioning region over which the integral (or expectation) is taken. Much of the proof appearing in the Appendix handles these correlations.

3.4. Overall Efficiency. Up to now, we considered average, or normalized, efficiency, where utility is averaged across market participants. The notion of normalized efficiency we have been studying is particularly useful when the designer is concerned with the expected outcomes of a clearinghouse's participants, or when contemplating individual incentives to shift from one institution to another (e.g., allowing for transfers or implementing an efficient rather than stable matching). However, market designers may also be concerned about overall efficiency. In this section, we study the wedge in terms of overall efficiency between optimal matchings (those maximizing overall efficiency) and stable matchings. In terms of overall efficiency, our results suggest a substantial welfare loss induced by stability, one that is more pronounced when preferences are idiosyncratic.

Formally, recall that we denoted by E_n the expected maximal utilitarian efficiency across all matchings. Our goal in this Section is to characterize the overall efficiency loss $L_n \equiv E_n - S_n$. In order to provide precise bounds on this difference, we focus on two polar cases in our setting: fully aligned and fully independent (or idiosyncratic) markets, where utilities are drawn from

¹⁹The proof appearing in the Online Appendix for this extreme case circumvents the formulas described here and utilizes more directly results from Pittel (1989).

the uniform distributions (the environments discussed in Sections 3.3 and 3.3, respectively).

We denote the efficiency loss associated with fully aligned markets (with uniformly distributed utilities over $[0, 1]$) with n participants on each side by L_n^A and the efficiency loss associated with fully independent, or idiosyncratic, markets (with uniformly distributed utilities over $[0, 1]$) by L_n^I . The following proposition provides bounds on L_n^A and L_n^I .

Proposition 3 [Efficiency Loss without Normalization].

1. For any $n \geq 3$,

$$\log n - 6 \leq L_n^A \leq 2 \log n.$$

2. The relative efficiency loss satisfies the following:

$$1 \leq \liminf_{n \rightarrow \infty} \frac{(\log n)^2 L_n^I}{n L_n^A} \leq 2.$$

The proposition illustrates the substantial welfare loss imposed by stability relative to any “optimal” matching, despite this loss having a vanishing effect on individual participants’ expected payoffs. The proposition also suggests that the structure of preferences impacts significantly the speed at which this welfare loss grows with market size, idiosyncratic preferences exhibiting a greater loss asymptotically. Namely, the ratio between the efficiency lost in idiosyncratic markets relative to that lost in aligned markets is asymptotically of the order of $n/(\log n)^2$, which increases with market size.

The proof of Proposition 3 relies on two sets of results. First, notice that Proposition 2 provides bounds on the speeds at which the expected efficiency of stable matchings grows for the environments we focus on here. We therefore need bounds on the speed with which the efficiency of optimal matchings grows. As it turns out, finding the maximal feasible efficiency is a variation of the *optimal assignment problem* in statistics. The literature on optimal assignment problems is still in flux and results are known only for particular distributions, mainly the uniform and exponential distributions. When preferences are fully aligned, we can interpret a result of Walkup (1979), which implies directly that when utilities are drawn from the uniform distribution, $2n - 6 \leq E_n \leq 2n$.²⁰

²⁰Follow-up work has improved upon this bound (see, for instance, Coppersmith and Sorkin, 1999, whose work suggests that $E_n \geq 2n - 3.88$). We use Walkup’s bound since it is sufficient for our conceptual message and as we use his method of proof to identify E_n when preferences are fully independent.

Consider now markets with fully independent preferences, drawn from the uniform distribution. That is, for each (i, j) , match utilities are given by u_{ij}^f and u_{ij}^w that are distributed uniformly on $[0, 1]$. We define $\tilde{u}_{ij} \equiv \frac{u_{ij}^f + u_{ij}^w}{2}$ and consider the maximal efficiency achieved by the optimal matching corresponding to a fully aligned market with preferences specified by $(\tilde{u}_{ij})_{i,j}$. Walkup (1979)'s result cannot be used directly, however, since now \tilde{u}_{ij} is distributed according to the symmetric triangular distribution over $[0, 1]$. In the Appendix, we modify the proof in Walkup (1979) and illustrate that, in this environment, $E_n \geq 2n - 3\sqrt{n}$. In fact, in the Online Appendix, we also show that $\lim_{n \rightarrow \infty} \frac{2n - E_n}{\sqrt{n}} \geq \sqrt{\frac{\pi}{2}}$ so that indeed the difference between E_n and $2n$ is of the order of \sqrt{n} .

4. ASSORTATIVE MARKETS

Another class of matching markets that plays an important role in many applications allows for some assortative preferences (see Becker, 1973). In such markets, one or both sides of the market agree on the ranking of the other side. For instance, medical residents may evaluate hospitals, at least to some extent, according to their publicly available rankings and hospitals may agree on the attributes that make a resident appealing (see Agarwal, 2015); similarly, potential adoptive parents may evaluate children up for adoption similarly (see Baccara et al., 2014); and so on. In this section, we illustrate that such markets, in which preferences are a combination of a common ranking across firms or workers and arbitrary idiosyncratic shocks, still entail asymptotically efficient stable matchings.

We assume that each agent has her own intrinsic value, which we denote by $(c_i^f)_{i=1}^n$ for firms and $(c_j^w)_{j=1}^n$ for workers. When firm f_i matches with worker w_j , the firm's utility is determined by the worker's intrinsic value c_j^w and the worker's value assessed individually by the firm, the idiosyncratic component z_{ij}^f . Similarly, worker w_j 's utility of matching with firm f_i is a combination of the firm's intrinsic value c_i^f and the worker's idiosyncratic assessment of the firm z_{ij}^w . That is,

$$\begin{aligned} u_{ij}^f &\equiv \Phi(c_j^w, z_{ij}^f) \quad \text{and} \\ u_{ij}^w &\equiv \Omega(c_i^f, z_{ij}^w). \end{aligned}$$

The functions $\Phi(., .)$ and $\Omega(., .)$ from \mathbb{R}_+^2 to \mathbb{R}_+ are continuous and strictly increasing in both

arguments. We assume that c_i^f , c_j^w , z_{ij}^f , and z_{ij}^w are all drawn independently from non-atomic continuous distributions over \mathbb{R}_+ , that all have bounded supports.

Let E_n^f and E_n^w be the expected maximal utilitarian welfare for n firms and workers, respectively, achievable by any market matching:

$$E_n^f \equiv \mathbb{E} \left(\max_{\mu \in M} \sum_{i=1}^n u_{i\mu(i)}^f \right) \quad \text{and} \quad E_n^w \equiv \mathbb{E} \left(\max_{\mu \in M} \sum_{i=1}^n u_{i\mu(i)}^w \right).$$

In the following proposition, we show that all stable matchings deliver approximately maximal utilitarian welfare as market size increases.

Proposition 4 [Efficiency of Stable Matchings]. *Stable matchings in assortative markets with idiosyncratic shocks are asymptotically efficient:*

$$\lim_{n \rightarrow \infty} \frac{E_n^f - S_n^f}{n} = \lim_{n \rightarrow \infty} \frac{E_n^w - S_n^w}{n} = 0.$$

The proof of Proposition 4 is a direct consequence of Lee (2017). Indeed, notice that it is without loss of generality to consider variables c_i^f , c_j^w , z_{ij}^f , and z_{ij}^w that are all uniformly distributed over $[0, 1]$, as we can always transform the aggregating utilities Φ and Ω monotonically in a consistent manner. Proposition 1 in the online appendix of Lee (2017) indicates that:

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\frac{S_n^f}{\sum_{i=1}^n \Phi(c_j^w, 1)} \right] = 1,$$

which, in turn, implies the claim of the Proposition pertaining to firms. A symmetric argument holds for the utilitarian efficiency experienced by workers.

Lee (2017) suggested that in settings such as these, for any stable matching mechanism, asymptotically, there is an “almost”-equilibrium that implements a stable matching corresponding to the underlying preferences. Formally, Lee (2017) implies that for any stable matching mechanism and any $\varepsilon, \delta, \theta > 0$, there exists N such that with probability of at least $1 - \delta$, a market of size $n > N$ has an ε -Nash equilibrium in which a fraction of at least $1 - \theta$ of agents reveal their true preferences. Together with our results, this suggests the following corollary.

Corollary 1 [Stable Matching Mechanisms]. *When preferences are hybrids of assortative and idiosyncratic components, stable matching mechanisms are asymptotically efficient and incentive compatible.*

In order to identify the speed of convergence, we restrict attention to linear functions Φ and Ω . For each pair (f_i, w_j) , we assume that:

$$\begin{aligned}\Phi(c_j^w, z_{ij}^f) &= (1 - \beta)c_j^w + \beta z_{ij}^f, \quad \text{and} \\ \Omega(c_i^f, z_{ij}^w) &= (1 - \beta)c_i^f + \beta z_{ij}^w,\end{aligned}$$

where $\beta \in [0, 1]$, and c_i^f , c_j^w , z_{ij}^f and z_{ij}^w are independently drawn from a uniform distribution over $[0, 1]$ for all i, j .

Notice that any matching generates the same expected utilitarian efficiency corresponding to the assortative component of preferences, evaluated were $\beta = 0$, given by $1/2$. Therefore, the maximum conceivable asymptotic efficiency is

$$\lim_{n \rightarrow \infty} E_n^f = \lim_{n \rightarrow \infty} E_n^w = (1 - \beta)\frac{1}{2} + \beta.$$

As mentioned, when $\beta = 0$, the expected utilitarian efficiency of the (generically unique) stable matching is $1/2$ for all n . When $\beta = 1$, our characterization in Proposition 2 provides the speed of convergence. The following proposition characterizes the speed at which efficiency per person converges to the maximum conceivable when $\beta \in (0, 1)$.

Proposition 5. *For any $\beta \in (0, 1)$,*

$$(1 - \beta)\frac{1}{2} + \beta - \frac{S_n^f}{n} = (1 - \beta)\frac{1}{2} + \beta - \frac{S_n^w}{n} = O(n^{-1/4}).$$

Proposition 5 suggests an important difference between markets entailing preferences that have an aligned component, relative to markets characterized by preferences with a dominant assortative component. The convergence speed in the latter is substantially faster.

Figure 3 corresponds to average efficiency levels in 100 simulated markets. As before, we depict a worst-case scenario, where we consider firms' average utilities when the worker-

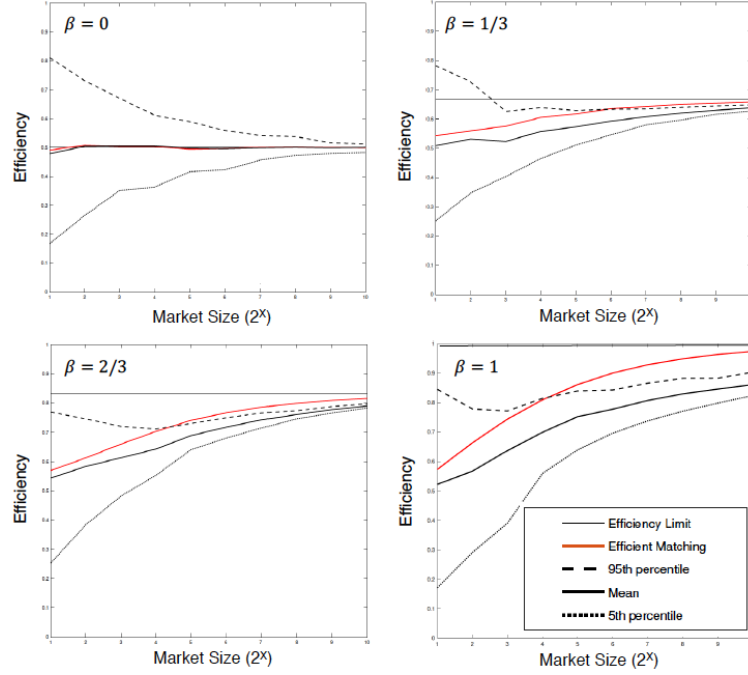


Figure 3: Convergence Speeds for Linear Hybrid Models of Assortative and Idiosyncratic Preferences (β is the Weight on the Idiosyncratic Component)

optimal stable matching is implemented in each market. We also depict the average maximal feasible efficiency. Notice that the maximal conceivable limit efficiency per person depends on β . In each panel of the figure, corresponding to a different level of β , we therefore also mark with a horizontal line that bound on the maximal conceivable limit efficiency. When markets contain about 1000 individuals on each side, the fraction of the maximal achievable efficiency that stable matchings yield is quite high, especially for $\beta \leq 2/3$. As for the case of aligned and idiosyncratic hybrid preferences, the dependence of that fraction on β is non-linear, with more pronounced efficiency losses appearing only for fairly high levels of β and for these fairly small markets.²¹

²¹The case considered here lends itself to particular match utilities that involve interaction terms. Consider a simple case with interactions, where for each firm i with attributes c_i^f and worker j with attributes c_j^w :

$$\begin{aligned} u_{ij}^f &= \beta c_i^f c_j^w + \varepsilon_{ij}^f \\ u_{ij}^w &= \beta c_i^f c_j^w + \varepsilon_{ij}^w, \end{aligned}$$

where c_i^f, c_j^w , and $\varepsilon_{ij}^f, \varepsilon_{ij}^w$ are random variables and we assume that individual attributes c_i^f and c_j^w have supports bounded above 0, say of the form $[a, 1]$, where $a > 0$. Then, when supports are bounded, it suffices

In purely assortative markets, all matchings of everyone in the market entail the same efficiency level. In particular, both normalized and overall efficiency levels are maximized under the stable matchings in such markets. In the Online Appendix, we also analyze the asymmetric case in which workers all share the same evaluation of firms with utilities determined uniformly, while firms have independent evaluations of workers. In that case, the speed of convergence of the efficiency of stable matchings converges to the maximum at a speed of $\frac{\log n}{n}$ and overall efficiency convergence patterns mimic those of fully aligned markets.

5. MARKETS WITH ASYMPTOTICALLY INEFFICIENT STABLE MATCHINGS

It is not very hard to find a large market where stable matchings are asymptotically utilitarian inefficient. Indeed, a replication of small markets in which stable matchings are utilitarian inefficient generates (larger) markets that entail stable matchings that are inefficient.²² However, in the setting we studied up to now, such markets are asymptotically unlikely. In this section, we study two classes of environments in which the probability of stable matchings not being utilitarian efficient remains substantial even when markets are large.

5.1. Severely Imbalanced Markets. Throughout the paper, we have assumed that markets are roughly balanced: our presentation pertained to coinciding volumes of firms and workers and, as mentioned at the outset, would carry through whenever the imbalance were bounded, e.g., if there were n firms (workers) and $n + k$ workers (firms), where k is fixed.²³

to consider the efficiency of stable matchings in a market with modified utilities that normalize each agent's original utility by her own attribute. Under further assumptions of distributions, these utilities are of the form we discuss here. We note that Menzel (2015) considers utility forms as such, where the set of attributes is finite and error terms follow extreme distributions, in the style of Dagsvik (2000). With those assumptions, Peski (2017) illustrates asymptotic efficiency in the roommate problem, which subsumes the marriage market.

²²One natural way to think of replicating an $m \times m$ market (such as the 2×2 market we first discussed in the introduction) characterized by utilities (u_{ij}^f, u_{ij}^w) is by considering a market of size $km \times km$, with match utilities $(\tilde{u}_{ij}^f, \tilde{u}_{ij}^w)$, where

$$\tilde{u}_{i'j'}^x = \begin{cases} u_{i' \bmod k, j' \bmod k}^x & i' \operatorname{div} k = j' \operatorname{div} k \\ 0 & \text{otherwise} \end{cases},$$

so that, for any $l = 0, 1, \dots, k - 1$, firms $f_{lm+1}, \dots, f_{(l+1)m}$ and workers $w_{lm+1}, \dots, w_{(l+1)m}$ have the same preferences over one another as in the original market, and generically prefer matching with agents in this 'sub-market' over matching with anyone else in the market.

²³In fact, the claims go through for any bounded difference in volumes – i.e., markets with n firms and $n + k(n)$ workers, where $k(n) \leq K$ for all n . The proofs for markets that entail idiosyncratic preference

Since in many real-world matching markets one side has more participants than the other, in this section, we study the robustness of our main result to the assumption that this imbalance is not too severe. This is particularly interesting in view of recent results by Ashlagi, Kanoria, and Leshno (2017) that illustrate the sensitivity of the structure of stable matchings to the relative sizes of both sides of the matching market.

We will assume that utilities from matching with anyone are positive almost always, whereas remaining unmatched generates zero utility. Under these assumptions, all participants of the scarce side of the market are generically matched in any stable matching. Furthermore, the Rural Hospital Theorem (see Roth and Sotomayor, 1992) assures that the set of unmatched individuals does not depend on the implemented stable matching. Since no matching can increase the number of matched individuals, a natural measure of efficiency considers the per-person expected utility, conditional on being matched. As before, since there might be multiple stable matchings, we will inspect the worst-case scenario.

We consider markets with n firms and $n + k(n)$ workers and examine the asymptotic efficiency for matched workers when the firm-optimal stable matching, the worker-pessimal stable matching, is implemented. We focus on cases in which the relative volumes of participants on both sides of the market are comparable, so that $\frac{k(n)}{n}$ is bounded.²⁴

Notice that the addition of workers can only improve firms' expected utility when focusing on the extremal stable matchings (see Roth and Sotomayor, 1992). Therefore, intuitively, in any balanced setting in which asymptotic efficiency is achieved, the introduction of more workers will maintain the asymptotic efficiency of stable matchings for firms.

The main result of this section is that asymptotic efficiency may break whenever markets are severely imbalanced *and* preferences exhibit substantial idiosyncratic components.

When markets are perfectly aligned or perfectly assortative, the proofs of Propositions 1

components need to be more carefully modified and are available from the authors. We note that the results are consistent with Ashlagi, Kanoria, and Leshno (2017). For instance, consider markets with n firms and $n + 1$ workers. Their results suggest that when markets are fully idiosyncratic, with high probability, in any stable matching, firms' average rank of employed workers is no more than $3 \log n$, whereas the workers' average rank of their employing firms is at least $n/3 \log n$. After normalization by n , both of these bounds converge to 0 as n becomes large.

²⁴We believe these are the cases that are interesting to consider from an applied perspective. Furthermore, whenever $\frac{k(n)}{n}$ explodes, the relevant efficiency statements would pertain to an insignificant fraction of firms that end up being matched.

and 4 carry through for arbitrary increasing functions $k(n)$ and asymptotic efficiency of stable matchings still holds.²⁵ We now focus on the case in which preferences are idiosyncratic, where we normalize the utility from remaining unmatched to be zero. Recall that S_n^w denotes the expected utilitarian efficiency experienced by workers in the worker-pessimal stable matching.

The following proposition illustrates the impacts of market imbalance. If one side of the market is proportionally larger and preferences are fully independent, inefficiency may arise even when markets are large.

Proposition 6 [Imbalanced Fully Independent Markets]. *Suppose $k(n) \geq \lambda n$ for some $\lambda > 0$, and all utilities $(u_{ij}^f)_{i,j}$ and $(u_{ij}^w)_{i,j}$ are independently drawn from the uniform distribution over $[0, 1]$. Then,*

$$\lim_{n \rightarrow \infty} \frac{S_n^w}{n} \leq \begin{cases} 1 - \frac{1}{-3 \log \lambda} & \text{for } 0 < \lambda \leq 1/2 \\ 1 - \frac{1}{3 \log 2} & \text{for } 1/2 < \lambda \end{cases}.$$

Notice that this indeed suggests inefficiency in large markets. For each realization of a market, characterized by realized utilities $(u_{ij}^f)_{i,j}$ and $(u_{ij}^w)_{i,j}$, consider the induced perfectly aligned market with utilities $(u_{ij}^* \equiv \frac{u_{ij}^f + u_{ij}^w}{2})_{i,j}$. That is, in the induced market, each matched firm and worker receive their average match utilities in the original market. The efficiency results pertaining to aligned markets then carry through for the induced market. Since these matchings produce the same per-participant utilities in the original markets, maximal efficiency can be achieved asymptotically. The wedge identified in Proposition 6 then implies a substantial asymptotic inefficiency generated by stability.

To gain some intuition on why severe imbalances can lead to inefficiencies when preferences are idiosyncratic, consider an extreme sequence of markets comprised of one firm and n workers. In such markets, the stable matching matches the firm with its favorite worker. Therefore, the firm's expected utility in the stable matching is the maximum of n samples from the uniform distribution over $[0, 1]$, which is $\frac{n}{n+1}$ and indeed converges to 1. However, since workers' utilities are drawn independently from the firm's, the worker matched under the stable matching has an expected utility of $\frac{1}{2}$. Nonetheless, for the matching that maximizes efficiency, we should look for j that maximizes $u_{1j}^f + u_{1j}^w$, which is distributed according

²⁵For markets with underlying preferences that are hybrids of assortative and idiosyncratic, slightly more involved arguments are required that follow directly from results in Lee (2017).

to the symmetric triangular distribution over $[0, 2]$. Therefore, the maximal feasible efficiency corresponds to the maximum of n samples from a symmetric triangular distribution over $[0, 2]$, which converges to 2 as n grows large. Roughly speaking, the crux of this example is that in stable matchings, the scarce side of the market does not take into account utilities achieved by the other side of the market. In particular, a matching that implies even a minuscule loss for the firm, but a substantial increase in the utility of the matched worker will not be implemented.

5.2. Sub-modularity in Match Qualities. Another important case in which inefficiency arises asymptotically pertains to assortative matching markets in which match utilities are sub-modular in partners' intrinsic values. For finite markets, Becker (1974) illustrated that sub-modularity in assortative markets leads to the negatively assortative utilitarian efficient matching and the positively assortative unique stable matching.

Formally, consider a sequence of $n \times n$ assortative markets in which firms' intrinsic values are given by $(c_i^f = \frac{i}{n})_{i=1}^n$ and workers' intrinsic values are given by $(c_j^w = \frac{j}{n})_{j=1}^n$. Match utilities are determined according to an "output function" ϕ :

$$u_{ij}^f = u_{ij}^w = \phi(c_i^f, c_j^w)$$

such that

$$\frac{\partial \phi(c_i^f, c_j^w)}{\partial c_i^f} > 0, \quad \frac{\partial \phi(c_i^f, c_j^w)}{\partial c_j^w} > 0.$$

The positively assortative matching partners each f_i with w_i , and it is the unique stable matching in these markets. The negatively assortative matching partners each f_i with w_{n+1-i} .

The cross-partial derivatives of the output function ϕ are crucial in determining whether the positively assortative matching is an efficient matching or not. Indeed, when ϕ is linear, all matchings generate the same utilitarian efficiency and both the positively and negatively assortative matchings are utilitarian efficient. When output is super-modular, $\frac{\partial^2 \phi(c_i^f, c_j^w)}{\partial c_i^f \partial c_j^w} > 0$, the positively assortative matching is an efficient matching, while when output is sub-modular, $\frac{\partial^2 \phi(c_i^f, c_j^w)}{\partial c_i^f \partial c_j^w} < 0$, the positively assortative matching (i.e., stable matching) is not an efficient matching, which is negatively assortative. Sub-modular preferences essentially imply that

marginal utilities with respect to a partner's intrinsic value are decreasing in one own's intrinsic value.

In order to illustrate how these features may carry through to large markets, we consider a particular class of output functions:

$$u_{ij}^f = u_{ij}^w = \phi(c_i^f, c_j^w) \equiv (c_i^f + c_j^w)^\alpha,$$

where $\alpha \in (0, 1)$, so that output is sub-modular.

Utilitarian welfare from the efficient matching (i.e., the negatively assortative matching) is

$$E_n = 2 \cdot \sum_{i=1}^n \left(\frac{i + \mu(i)}{n} \right)^\alpha = 2 \cdot \sum_{i=1}^n \left(\frac{n+1}{n} \right)^\alpha = 2n \left(\frac{n+1}{n} \right)^\alpha.$$

Therefore,

$$\lim_{n \rightarrow \infty} \frac{E_n}{2n} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^\alpha = 1.$$

On the other hand, utilitarian welfare from the stable matching (i.e., the positively assortative matching) is

$$S_n = 2 \cdot \sum_{i=1}^n \left(\frac{i + \mu(i)}{n} \right)^\alpha = 2 \cdot \sum_{i=1}^n \left(\frac{2i}{n} \right)^\alpha.$$

Note that

$$\frac{1}{n} \sum_{i=1}^n \left(\frac{2i}{n} \right)^\alpha \geq \int_0^1 (2x)^\alpha dx = \frac{2^\alpha}{\alpha+1} \geq \frac{1}{n} \sum_{i=0}^{n-1} \left(\frac{2i}{n} \right)^\alpha = \frac{1}{n} \sum_{i=1}^n \left(\frac{2i}{n} \right)^\alpha - \frac{2^\alpha}{n}.$$

Thus,

$$\lim_{n \rightarrow \infty} \frac{S_n}{2n} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left(\frac{2i}{n} \right)^\alpha = \frac{2^\alpha}{\alpha+1}.$$

When α is close to 0, output is very insensitive to individual abilities, and any matching, in particular the stable one, generates efficiency close to the optimum. When α is close to 1, the output function is “almost linear” in individual abilities and, again, the stable matching is asymptotically nearly efficient. Nonetheless, for intermediate values of α , asymptotic efficiency is bounded strictly below 1, which is achieved by the most efficient matching.

6. CONCLUSIONS

This paper illustrates that for a large class of preference distributions, stable matchings are asymptotically efficient, at least when considering expected efficiency across market participants. In these environments transfers are not necessary to implement efficient outcomes. This is particularly relevant in view of the observation that many markets entail fixed wages (see Hall and Kreuger, 2012), or are subject to legal or “moral” constraints that ban transfers (see Roth, 2007). In fact, in such settings, a market designer faces no trade-off between efficiency and stability when the market is large, and the objective of stability is viable even on efficiency grounds.

Our results also illustrate the speeds of convergence of the efficiency of stable matchings to the optimum. Idiosyncratic preferences yield a substantially lower speed of convergence than those exhibited in markets with aligned or assortative preferences. This suggests that market designers concerned with efficiency should consider market size with special caution.

Markets with idiosyncratic preferences are also fragile to imbalances in the volumes of participants on either side. When those imbalances are severe—when the volume of one side constitutes a fixed fraction of the volume of the other side—stable matchings are no longer efficient in general, even when markets are large, and transfers could prove beneficial.

Another important caveat regards the notion of efficiency one uses. Indeed, if one focuses on overall efficiency, rather than that averaged across participants, stability and efficiency display a pronounced wedge across all of the preference constellations we consider (hybrids of assortative, idiosyncratic, and assortative preferences). An approach aiming at maximizing overall efficiency would then potentially require mechanisms that are not necessarily stable or ones allowing transfers.

While our results simply assess the efficiency features of stable matchings in a variety of markets, they open the door for many interesting questions regarding incentive compatibility of efficient mechanisms. When preferences combine assortative and idiosyncratic components, stable matchings are not only asymptotically efficient, they are also asymptotically incentive compatible. Our results then serve as a rather positive defense of commonly used mechanisms such as the Gale-Shapley (1962) deferred acceptance algorithm—in such settings, they are asymptotically incentive compatible and efficient (our Corollary 1). However, for a designer

concerned with overall efficiency, it would be important to analyze the most efficient incentive compatible mechanisms. Furthermore, for other types of preferences, even the question of incentive compatibility of stable mechanisms in large markets is still open.

Our analysis pertains to one-to-one matching markets such as those matching doctors and residency positions, rabbis and congregations, children up for adoption and potential adoptive parents, etc. Expanding the analysis to many-to-one matching markets would be an interesting direction for future research as well. A leading application would be school choice. Additional details could be relevant for such an analysis, for instance parents' investments in student qualifications (see Cole, Mailath, and Postlewaite, 2001) and the interaction between matching processes and the real estate market (see Avery and Pathak, 2015). Naturally, incorporating these details would make welfare assessments more subtle in this context.

7. APPENDIX – PROOFS

7.1. Proof of Propositions 1 and 2 for Full Alignment. We start by proving Propositions 1 and 2 for the case in which preferences are fully aligned, as in Section 3.3. That is, the match utilities of each firm and worker pair depend only on a random common value. Formally, we assume the functions $\phi(\cdot, \cdot)$ and $\omega(\cdot, \cdot)$ are both strictly increasing in c_{ij} but independent of the idiosyncratic components.

We begin with the derivation of a formula for S_n suggested in the text when utilities are distributed uniformly and illustrate both asymptotic efficiency and the speed of convergence for that case. We then generalize our asymptotic efficiency result to arbitrary continuous distributions.

As illustrated in the text, realized utilities $(u_{ij})_{i,j}$ generically induce a unique stable matching. Denote by $u_{[k;n]}$ the k -th highest match utility of pairs matched within that unique stable matching. Therefore,

$$\frac{S_n}{2} = \mathbb{E} \left(\sum_{k=1}^n u_{[k;n]} \right) = \sum_{k=1}^n \mathbb{E}(u_{[k;n]}).$$

We use induction to show that for $k = 1, \dots, n$,

$$\mathbb{E}(u_{[k;n]}) = \frac{n^2}{n^2 + 1} \frac{(n-1)^2}{(n-1)^2 + 1} \cdots \frac{(n-k+1)^2}{(n-k+1)^2 + 1}.$$

For $k = 1$, $u_{[1;n]}$ is the maximal utility achievable from all firm-worker pairs. Thus, $u_{[1;n]}$ is the highest entry from n^2 samples from the uniform distribution over $[0, 1]$ and so:

$$\mathbb{E}(u_{[1;n]}) = \frac{n^2}{n^2 + 1}$$

Suppose the claim is shown for $k-1$. From the construction of the stable matching, $u_{[k;n]}$ is the maximal utility among all firm and worker pairs, after all firms and workers receiving the $k-1$ highest utilities within the stable matching have been removed from the market. Thus, $u_{[k;n]}$ is the highest entry from $(n-k+1)^2$ samples from the uniform distribution over $[0, 1]$ restricted so that each sample has a value lower than or equal to $u_{[k-1;n]}$. Therefore,

$$\mathbb{E}(u_{[k;n]} | u_{[k-1;n]}) = u_{[k-1;n]} \frac{(n-k+1)^2}{(n-k+1)^2 + 1}.$$

By the law of iterated expectations,

$$\begin{aligned}\mathbb{E}(u_{[k;n]}) &= \mathbb{E}(\mathbb{E}(u_{[k;n]}|u_{[k-1;n]})) = \mathbb{E}(u_{[k-1;n]}) \frac{(n-k+1)^2}{(n-k+1)^2+1} \\ &= \frac{n^2}{n^2+1} \cdots \frac{(n-k+2)^2}{(n-k+2)^2+1} \frac{(n-k+1)^2}{(n-k+1)^2+1},\end{aligned}$$

where the last equality is from the induction hypothesis. The formula for S_n follows.

We now turn to the proof of Propositions 1 and 2 for this setting. First, denote by $C_n = 1 - \frac{S_n}{2n}$.

Notice that $C_1 = 1/2$. For any $n > 1$,

$$\frac{S_n}{2} = \frac{n^2}{n^2+1} + \frac{n^2}{n^2+1} \frac{S_{n-1}}{2},$$

which implies that

$$\begin{aligned}C_n &= 1 - \frac{S_n}{2n} = 1 - \frac{n}{n^2+1} - \frac{n}{n^2+1} \frac{S_{n-1}}{2} \\ &= \frac{1}{n^2+1} + \frac{n(n-1)}{n^2+1} \left(1 - \frac{S_{n-1}}{2(n-1)}\right) = \frac{1}{n^2+1} + \frac{n(n-1)}{n^2+1} C_{n-1}.\end{aligned}$$

Then, we can find that

$$C_3 = \frac{1}{10} + \frac{6}{10} C_2 = \frac{1}{10} + \frac{6}{10} \left(\frac{1}{5} + \frac{21}{5} \frac{1}{2}\right) = \frac{17}{50} = 0.34,$$

which is between $\frac{1}{2} \frac{\log 3}{3} = 0.18\dots$ and $\frac{\log 3}{3} = 0.37\dots$

For any $n > 3$, suppose

$$\frac{1}{2} \frac{\log(n-1)}{n-1} \leq C_{n-1} \leq \frac{\log(n-1)}{n-1}.$$

Then,

$$\frac{1}{n^2+1} + \frac{n}{2(n^2+1)} \log(n-1) \leq C_n = \frac{1}{n^2+1} + \frac{n(n-1)}{n^2+1} C_{n-1} \leq \frac{1}{n^2+1} + \frac{n}{n^2+1} \log(n-1).$$

Notice that

$$\frac{1}{n^2 + 1} + \frac{n}{n^2 + 1} \log(n - 1) < \frac{1}{n^2} + \frac{1}{n} \log(n - 1) < \frac{\log n}{n},$$

where the second inequality, which is equivalent to $\log\left(\frac{n-1}{n}\right) < -\frac{1}{n}$, holds from $\frac{n-1}{n} = 1 - \frac{1}{n} < e^{-1/n}$.

Further, we have

$$\frac{1}{n^2 + 1} + \frac{n}{2(n^2 + 1)} \log(n - 1) = \frac{2 + n \log(n - 1)}{2(n^2 + 1)} > \frac{n \log n + (1/2)}{2(n^2 + 1)} > \frac{\log n}{2n},$$

where the first inequality, which is equivalent to $n \log\left(\frac{n}{n-1}\right) < \frac{3}{2}$, holds for $n = 4$ since $4 \log(4/3) = 1.15 \dots$ and for $n > 4$ since

$$\frac{d\left(n \log\left(\frac{n}{n-1}\right)\right)}{dn} = \log\left(\frac{n}{n-1}\right) - \frac{1}{n-1} < 0$$

as $\frac{n}{n-1} = 1 + \frac{1}{n-1} < e^{\frac{1}{n-1}}$ holds for $n \geq 3$.

The proof of Propositions 1 and 2 for this case then follows.

We now show that asymptotic efficiency of stable matchings holds even when the match utilities of firm and worker pairs are independently and identically drawn from an arbitrary distribution G over $[0, 1]$. Indeed, we show that for any $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{S_n}{2n} \geq 1 - \varepsilon.$$

We construct a uniform distribution G' such that G first order stochastically dominates G' . The support of G is $[0, 1]$, so there exists γ such that $1 - \varepsilon < \gamma < 1$ and $G(\gamma) < 1$.

Let

$$G'(x) = \frac{1 - G(\gamma)}{\gamma} x + G(\gamma),$$

so that G' is the uniform distribution over $[0, \gamma]$.

Let S'_n be the expected utilitarian efficiency derived from the stable matching when utilities

are drawn from $G'(x)$. The proof above can be replicated to show

$$\lim_{n \rightarrow \infty} \frac{S'_n}{2n} = \gamma.$$

Note that G first order stochastically dominates G' . Thus, every order statistic corresponding to samples from G first order stochastically dominates the corresponding order statistic of the same number of samples from G' . For every realized utilities $(u_{ij})_{i,j}$, S_n is a sum of specific n order statistics. Thus, the utilitarian efficiency from the stable matching under G first order stochastically dominates that under G' . It follows that

$$\lim_{n \rightarrow \infty} \frac{S_n}{2n} \geq \lim_{n \rightarrow \infty} \frac{S'_n}{2n} = \gamma > 1 - \varepsilon.$$

Since ε is arbitrary, $\lim_{n \rightarrow \infty} \frac{S_n}{2n} = 1$, as desired. ■

7.2. Proof of Proposition 1 for Aligned Preferences with Idiosyncratic Shocks.

We now provide the proof of Proposition 1 for the case in which firms' and workers' match utilities, $\phi(.,.)$ and $\omega(.,.)$, are strictly increasing in the idiosyncratic components, z_{ij}^f and z_{ij}^w and either both are independent of the common component c_{ij} or both are strictly increasing in c_{ij} . It is without loss of generality to consider c_{ij} , z_{ij}^f , and z_{ij}^w that are all uniformly distributed over $[0, 1]$. Indeed, an appropriate change of variables (or, equivalently, a monotone transformation of utilities), will generate an equivalent setting in which the underlying distributions are uniform.²⁶

The model is potentially a mixture of aligned preferences captured by the variables $c = (c_{ij})_{i,j}$ and independent preferences captured by the variables $z^f = (z_{ij}^f)_{i,j}$ and $z^w = (z_{ij}^w)_{i,j}$. Accordingly, our proof is comprised of two parts.

For each realization (c, z^f, z^w) , let

$$\bar{F}(\varepsilon; c, z^f, z^w) \equiv \{f_i | c_{i\mu^w(i)} \leq 1 - \varepsilon\}.$$

Whenever $\phi(.,.)$ and $\omega(.,.)$ are strictly increasing in the common component, we first show

²⁶See the online appendix of Lee (2017) for details.

that for any $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\frac{|\bar{F}(\varepsilon; c, z^f, z^w)|}{n} \right] = 0. \quad (2)$$

In the second part of the proof, we show that for any $\varepsilon > 0$,

$$P \left(\frac{\sum_{i=1}^n z_{i\mu^w}^f}{n} \leq 1 - \varepsilon \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3)$$

Proposition 1, for cases excluding fully aligned preferences, is immediate from (2) and (3). For any ε , there exist ε' such that if $\phi(c, z^f) < \phi(1, 1) - \varepsilon$ then either $c < 1 - \varepsilon'$ or $z^f < 1 - \varepsilon'$. Therefore,

$$\frac{1}{n} |\{f_i | u_{i\mu^w}(i) \leq \phi(1, 1) - \varepsilon\}| \leq \frac{1}{n} |\{f_i | c_{i\mu^w}(i) \leq 1 - \varepsilon'\}| + \frac{1}{n} |\{f_i | z_{i\mu^w}(i) \leq 1 - \varepsilon'\}|.$$

The right hand side converges to zero in probability by (2) and (3).

Proof of Equation (2). Assume that $\phi(., .)$ and $\omega(., .)$ are strictly increasing in the common component c_{ij} , in which case Equation (2) is relevant for our proof.

A **graph** G is a pair (V, E) , where V is a set called **nodes** and E is a set of unordered pairs (i, j) or (j, i) of $i, j \in V$ called **edges**. The nodes i and j are called the **endpoints** of (i, j) . We say that a graph $G = (V, E)$ is **bipartite** if its node set V can be partitioned into two disjoint subsets V_1 and V_2 such that each of its edges has one endpoint in V_1 and the other in V_2 .

A **biclique** of a bipartite graph $G = (V_1 \cup V_2, E)$ is a set of nodes $U_1 \cup U_2$ such that $U_1 \subset V_1$, $U_2 \subset V_2$, and for all $i \in U_1$ and $j \in U_2$, $(i, j) \in E$. In other words, a biclique is a complete bipartite subgraph of G . We say that a biclique is **balanced** if $|U_1| = |U_2|$, and refer to a balanced biclique with the maximal number of nodes as a **maximal balanced biclique**.

Given a partitioned set $V_1 \cup V_2$, we consider a random bipartite graph $G(V_1 \cup V_2, p)$. A bipartite graph $G = (V_1 \cup V_2, E)$ is constructed so that each pair of nodes, one in V_1 and the other in V_2 , is included in E independently with probability p . We use the following proposition in the proof.

Proposition 7 [Dawande et al., 2001]. *Consider a random bipartite graph $G(V_1 \cup V_2, p)$, where $0 < p < 1$ is a constant, $|V_1| = |V_2| = n$, and $\delta(n) = \log n / \log \frac{1}{p}$. If a maximal balanced bi-*

clique of this graph has size $B \times B$, then

$$Pr(\delta(n) \leq B \leq 2\delta(n)) \rightarrow 1, \quad \text{as } n \rightarrow \infty.$$

By continuity of $\phi(.,.)$ and $\omega(.,.)$, there exists $\varepsilon' > 0$ such that

$$c_{ij}, z_{ij}^f, z_{ij}^w > 1 - \varepsilon' \implies \phi(c_{ij}, z_{ij}^f) > \phi(1 - \varepsilon, 1) \text{ and } \omega(c_{ij}, z_{ij}^w) > \omega(1 - \varepsilon, 1).$$

For each realization (c, z^f, z^w) , we draw a bipartite graph such that $F \cup W$ is the set of nodes (where F and W constitute the two parts of the graph), and each pair of f_i and w_j is connected by an edge if and only if at least one of c_{ij} , z_{ij}^f , or z_{ij}^w is lower than or equal to $1 - \varepsilon'$.

Let

$$\bar{W}(\varepsilon; c, z^f, z^w) \equiv \{w_j | \mu^w(j) \in \bar{F}(\varepsilon; c, z^f, z^w)\}.$$

Then $\bar{F} \cup \bar{W}$ is a balanced biclique. If a pair (f_i, w_j) from $\bar{F} \cup \bar{W}$ is not connected by an edge, then the pair can achieve utilities $\phi(c_{ij}, z_{ij}^f) > \phi(1 - \varepsilon, 1)$ and $\omega(c_{ij}, z_{ij}^w) > \omega(1 - \varepsilon, 1)$ because $c_{ij}, z_{ij}^f, z_{ij}^w > 1 - \varepsilon'$. The two utilities are higher than their utilities under μ^w . This contradicts μ^w being stable.

Proposition 7 then implies equation (2).

Proof of Equation (3). Let $\mu \equiv \{(i, i) | i = 1, \dots, n\}$. By symmetry, each one of the $n!$ matchings has the same probability of being both stable and entailing a sum of firms' idiosyncratic components that is lower than or equal to $(1 - \varepsilon)n$. Therefore,

$$P\left(\frac{\sum_{i=1}^n z_{i\mu(i)}^f}{n} \leq 1 - \varepsilon\right) \leq n!P\left(\mu \text{ is stable and } \frac{\sum_{i=1}^n z_{i\mu(i)}^f}{n} \leq 1 - \varepsilon\right).$$

For each realization (c, z^f, z^w) , we consider the following profile of utilities.

$$\begin{aligned} \tilde{u}_{ij}^f &= \phi(c_{ij}, z_{ij}^f) \quad \text{and} \quad \tilde{u}_{ij}^w = \omega(c_{ij}, z_{ij}^w) \quad \text{if } i \neq j, \quad \text{and} \\ \tilde{u}_{ij}^f &= \phi(1, z_{ij}^f) \quad \text{and} \quad \tilde{u}_{ij}^w = \omega(c_{ij}, z_{ij}^w) \quad \text{if } i = j. \end{aligned}$$

Remark 1. *If preferences are fully idiosyncratic, so that ϕ and ω are independent of c_{ij} , then $\tilde{u}_{ij} = u_{ij}$ for all $f_i \in F$ and $w_j \in W$.*

If a pair of f_i and w_j with $i \neq j$ is a blocking pair of μ under match utilities $(\tilde{u}_{ij}^f, \tilde{u}_{ij}^w)$, then the pair also blocks μ under the actual realized utilities. Thus,

$$P\left(\mu \text{ is stable and } \frac{\sum_{i=1}^n z_{i\mu(i)}^f}{n} \leq 1 - \varepsilon\right) \leq P_{\varepsilon,n}$$

where $P_{\varepsilon,n}$ is the probability that μ is stable with respect to the utilities $(\tilde{u}_{ij}^f, \tilde{u}_{ij}^w)_{i,j}$ and $\frac{\sum_{i=1}^n z_{i\mu(i)}^f}{n} \leq 1 - \varepsilon$. We prove that $n!P_{\varepsilon,n}$ converges to zero as n increases.

Preparation Steps. We denote by Γ^f the marginal distribution of \tilde{u}_{ij}^f for pairs of (f_i, w_j) that are not matched under μ (i.e., $i \neq j$), and by Γ^w the marginal distribution of \tilde{u}_{ij}^w for any pair of (f_i, w_j) . We define, for all i, j , $\hat{u}_{ij}^f = \Gamma^f(\tilde{u}_{ij}^f)$ and $\hat{u}_{ij}^w = \Gamma^w(\tilde{u}_{ij}^w)$.

Remark 2. *The marginal distributions of \hat{u}_{ij}^f for firm and worker pairs with $i \neq j$, and \hat{u}_{ij}^w for all pairs are uniform over $[0, 1]$. Whereas, for pairs with $i = j$, the marginal distribution of \hat{u}_{ij}^f first order stochastically dominates the uniform distribution over $[0, 1]$.*

For each given realization $(\hat{u}_{ii}^f)_{i=1}^n$ and $(\hat{u}_{jj}^w)_{j=1}^n$, the probability that μ is stable with respect to $(\tilde{u}_{ij}^f, \tilde{u}_{ij}^w)_{i,j}$ is the same as

$$\Pi_{1 \leq i \neq j \leq n} \left(1 - P[\hat{u}_{ij}^f > \hat{u}_{ii}^f \text{ and } \hat{u}_{ij}^w > \hat{u}_{jj}^w]\right).$$

Note that c_{ij} , z_{ij}^f , and z_{ij}^w are independently and identically distributed, so they are positively associated (See Theorem 2.1 in Esary, Proschan, and Walkup, 1967). Indeed, since $\Gamma^f(\phi(\cdot, \cdot))$ and $\Gamma^w(\omega(\cdot, \cdot))$ are non-decreasing functions of c_{ij} , z_{ij}^f , and z_{ij}^w , the covariance of the corresponding values, \hat{u}_{ij}^f and \hat{u}_{ij}^w for $i \neq j$ is non-negative.

Thus, we have

$$P[\hat{u}_{ij}^f > \hat{u}_{ii}^f \text{ and } \hat{u}_{ij}^w > \hat{u}_{jj}^w] \geq P[\hat{u}_{ij}^f > \hat{u}_{ii}^f]P[\hat{u}_{ij}^w > \hat{u}_{jj}^w] = (1 - \hat{u}_{ii}^f)(1 - \hat{u}_{jj}^w).$$

Last, take any $0 < \gamma < 1/2$ such that

$$\Gamma^f(\phi(1, z_{ii}^f)) \leq (1 - \gamma) + \gamma z_{ii}^f.$$

Then,

$$1 - \hat{u}_{ii}^f \geq \gamma(1 - z_{ii}^f).$$

Therefore, for each realization of $c_{ij}, z_{ij}^f, z_{ij}^w$ for pairs with $i = j$, the probability that μ is stable is bounded above by

$$\prod_{1 \leq i \neq j \leq n} \left(1 - \gamma(1 - z_{ii}^f)(1 - \hat{u}_{jj}^w) \right).$$

We therefore obtain that

$$P_{\varepsilon, n} \leq \int \int_{\sum_{i=1}^n z_{ii}^f \leq (1-\varepsilon)n} \prod_{1 \leq i \neq j \leq n} \left(1 - \gamma(1 - z_{ii}^f)(1 - \hat{u}_{jj}^w) \right) d\mathbf{z}_{\mathbf{ii}}^f d\hat{\mathbf{u}}_{\mathbf{jj}}^w.$$

Now, let $x_i = 1 - z_{ii}^f$ and $y_j = 1 - \hat{u}_{jj}^w$. Then,

$$P_{\varepsilon, n} \leq \int_{\substack{\mathbf{0} \leq \mathbf{x}, \mathbf{y} \leq 1 \\ \varepsilon n \leq \sum_{i=1}^n x_i}} \prod_{1 \leq i \neq j \leq n} (1 - \gamma x_i y_j) d(\mathbf{x}, \mathbf{y}).$$

Proof of Convergence.

$$\begin{aligned} P_{\varepsilon, n} &\leq \int_{\substack{\mathbf{0} \leq \mathbf{x}, \mathbf{y} \leq 1 \\ \varepsilon n \leq \sum_{i=1}^n x_i}} \prod_{1 \leq i \neq j \leq n} (1 - \gamma x_i y_j) d(\mathbf{x}, \mathbf{y}) \\ &= \int_{\substack{\mathbf{0} \leq \mathbf{x}, \mathbf{y} \leq 1 \\ \varepsilon n \leq \sum_{i=1}^n x_i}} \prod_{1 \leq j \leq n} \underbrace{\left(\int_0^1 \prod_{i \neq j} (1 - \gamma x_i y_j) dy_j \right)}_{(*)} d\mathbf{x}. \end{aligned}$$

Let $t = n^{-7/8}$ and $\delta = e^t(1 - \gamma t)$. As $0 < \gamma < 1/2$, for any $0 \leq z \leq t$, we have $1 - \gamma z \leq \delta e^{-z}$. Thus, when $0 \leq y_j \leq t$, we have $0 \leq x_i y_j \leq t$, so $1 - \gamma x_i y_j \leq \delta \exp(-x_i y_j)$. In addition, $1 + \gamma z \leq e^{\gamma z}$ for any z , so $1 - \gamma x_i y_j \leq \exp(-\gamma x_i y_j)$.

Therefore,

$$\begin{aligned}
(*) &= \int_0^t \prod_{i \neq j} (1 - \gamma x_i y_j) dy_j + \int_t^1 \prod_{i \neq j} (1 - \gamma x_i y_j) dy_j \\
&= \int_0^t \prod_{i \neq j} \delta \exp(-x_i y_j) dy_j + \int_t^1 \prod_{i \neq j} \exp(-\gamma x_i y_j) dy_j \\
&= \delta \int_0^t \exp\left(-y_j \sum_{i \neq j} x_i\right) dy_j + \int_t^1 \exp\left(-\gamma y_j \sum_{i \neq j} x_i\right) dy_j
\end{aligned}$$

Let

$$s = \sum_{i=1}^n x_i \quad \text{and} \quad s_j = \sum_{i \neq j} x_i.$$

Then,

$$\begin{aligned}
(*) &= \delta \int_0^t \exp(-y_j s_j) dy_j + \int_t^1 \exp(-\gamma y_j s_j) dy_j \\
&= \delta \frac{1 - e^{-ts_j}}{s_j} + \frac{e^{-\gamma ts_j} - e^{-\gamma s_j}}{\gamma s_j} \leq \frac{1}{s_j} \left(\delta + \frac{1}{\gamma} \exp(-\gamma ts_j) \right)
\end{aligned}$$

We claim that

$$\delta + \frac{1}{\gamma} \exp(-\gamma ts_j) < \exp(n^{-6/7}) \quad (4)$$

for sufficiently large n .

As $s > \varepsilon n$, we have $s_j > \varepsilon n - 1$. Thus, (4) follows from

$$\exp(n^{-7/8}) + \frac{1}{\gamma} \exp(-\gamma n^{-7/8}(\varepsilon n - 1)) < \exp(n^{-6/7}),$$

for any sufficiently large n .

Now, we have

$$\begin{aligned}
P_{\varepsilon, n} &\leq \int_{\substack{\mathbf{0} \leq \mathbf{x} \leq 1 \\ \varepsilon n \leq \sum_{i=1}^n x_i}} \prod_{1 \leq j \leq n} (*) d\mathbf{x} \\
&\leq \int_{\substack{\mathbf{0} \leq \mathbf{x} \leq 1 \\ \varepsilon n \leq \sum_{i=1}^n x_i}} \prod_{1 \leq j \leq n} \left(\frac{1}{s_j} \exp(n^{-6/7}) \right) d\mathbf{x} = \exp(n^{1/7}) \int_{\substack{\mathbf{0} \leq \mathbf{x} \leq 1 \\ \varepsilon n \leq \sum_{i=1}^n x_i}} \prod_{1 \leq j \leq n} \left(\frac{1}{s_j} \right) d\mathbf{x}.
\end{aligned}$$

Note that $\left(\log \frac{1}{s_j}\right)' = -\frac{1}{s_j}$. Thus,

$$\sum_{j=1}^n \log \frac{1}{s_j} = \sum_{j=1}^n \left(\log \frac{1}{s} + \log \frac{s}{s_j} \right) = n \log \frac{1}{s} + \sum_{j=1}^n \log \frac{s}{s - x_j}.$$

In the last term,

$$\frac{s}{s - x_j} \leq \frac{s}{s - 1} \leq \frac{\varepsilon n}{\varepsilon n - 1}, \quad \text{for any sufficiently large } n.$$

Thus,

$$\sum_{j=1}^n \log \frac{1}{s_j} \leq n \ln \frac{1}{s} + n \log \frac{\varepsilon n}{\varepsilon n - 1}.$$

Moreover

$$n \log \frac{\varepsilon n}{\varepsilon n - 1} = \log \left(\left(1 + \frac{1}{\varepsilon n - 1} \right)^n \right) \rightarrow \frac{1}{\varepsilon} \quad \text{as } n \rightarrow \infty,$$

which implies that for any $c > \frac{1}{\varepsilon}$,

$$\sum_{j=1}^n \log \frac{1}{s_j} \leq n \log \frac{1}{s} + c, \quad \text{for any sufficiently large } n.$$

Therefore,

$$n! P_{\varepsilon, n} \leq n! \exp(n^{1/7}) \int_{\varepsilon n \leq s} \exp \left(n \log \frac{1}{s} + c \right) f_n(s) ds,$$

where $f_n(s)$ is the probability distribution function of s .

We show the convergence of the right hand side of the above inequality by using the following Lemma.²⁷

Lemma 1 [Pittel, 1989]. *Let x_1, \dots, x_{n-1} be i.i.d samples from the Uniform distribution over $[0, 1]$. Denote by $x_{(k)}$ the k 'th highest of these samples. We define a random variable*

$$r_n = \max_{0 \leq i \leq n-1} \{x_{(i)} - x_{(i+1)}\},$$

where $x_{(0)} \equiv 1$ and $x_{(n)} \equiv 0$.

²⁷The Lemma follows from Lemma 1 combined with the first two equations on the top of page 548 in Pittel (1989).

Then,

$$f_n(s) = \frac{s^{n-1}}{(n-1)!} Pr(r_n \leq s^{-1}),$$

and

$$Pr(r_n \leq x) \leq \exp\left(-ne^{-x(n+n^{9/14})}\right) + O\left(e^{-\frac{n^{2/7}}{2}}\right).$$

By applying Lemma 1, we get

$$\begin{aligned} n!P_{\varepsilon,n} &\leq n! \exp(n^{1/7}) \int_{\varepsilon n \leq s} \exp\left(n \log \frac{1}{s} + c\right) \frac{s^{n-1}}{(n-1)!} Pr(r_n \leq s^{-1}) ds \\ &\leq e^c n \exp(n^{1/7}) Pr\left(r_n \leq \frac{1}{\varepsilon n}\right) \int_{\varepsilon n}^n \frac{1}{s} ds \\ &= e^c n \exp(n^{1/7}) Pr\left(r_n \leq \frac{1}{\varepsilon n}\right) (-\log \varepsilon). \end{aligned}$$

Now,

$$\begin{aligned} n \exp(n^{1/7}) Pr\left(r_n \leq \frac{1}{\varepsilon n}\right) &\leq n \exp(n^{1/7}) \left(\exp\left(-ne^{-\frac{1+n-\frac{5}{14}}{\varepsilon}}\right) + O\left(e^{-\frac{n^{2/7}}{2}}\right) \right) \\ &= \exp\left(\log n + n^{1/7} - ne^{-\frac{1+n-\frac{5}{14}}{\varepsilon}}\right) + O\left(n \exp(n^{1/7} - \frac{1}{2}n^{2/7})\right). \end{aligned}$$

Both of the last two terms converge to 0. This completes the proof. \blacksquare

7.3. Proof of Proposition 2. We focus here on the case in which match utilities depend non-trivially on the idiosyncratic components, i.e. when $\alpha > 0$. The case where utilities are fully aligned, $\alpha = 0$, was shown in Section 7.1 in this Appendix.

The model is a mixture of aligned preferences and independent preferences. As such, our proof is comprised of two parts.

Fix $\varepsilon_n = \frac{2}{\log n}$.

For each market realization (c, z^f, z^w) , let

$$\bar{F}(\varepsilon_n; c, z^f, z^w) \equiv \{f_i | c_{i\mu^w(i)} \leq 1 - \varepsilon_n\}.$$

Whenever $\alpha < 1$, so that utilities depend non-trivially on the common component, we first

show that

$$\eta_n \equiv \mathbb{E} \left[\frac{|\bar{F}(\varepsilon_n; c, z^f, z^w)|}{n} \right] = o(n^{-1/2}) \quad \text{as } n \rightarrow \infty. \quad (5)$$

In the second part of the proof, we show that

$$\zeta_n \equiv P \left(\frac{\sum_{i=1}^n z_{i\mu^w}^f}{n} \leq 1 - \varepsilon_n \right) = o(e^{-n^{1/4}}) \quad \text{as } n \rightarrow \infty. \quad (6)$$

Proposition 2 is immediate from (5) and (6). Note that

$$\begin{aligned} 1 - \frac{S_n^f}{n} &= \mathbb{E} \left[\frac{\sum_{i=1}^n (1 - u_{i\mu^w})}{n} \right] \\ &= (1 - \alpha) \mathbb{E} \left[\frac{\sum_{i=1}^n (1 - c_{i\mu^w})}{n} \right] + \alpha \mathbb{E} \left[\frac{\sum_{i=1}^n (1 - z_{i\mu^w}^f)}{n} \right] \\ &\leq (1 - \alpha)(\eta_n + \varepsilon_n(1 - \eta_n)) + \alpha(\zeta_n + \varepsilon_n(1 - \zeta_n)) \\ &\leq \varepsilon_n + (1 - \alpha)\eta_n + \alpha\zeta_n. \end{aligned}$$

Therefore,

$$\limsup_{n \rightarrow \infty} \left(1 - \frac{S_n^f}{n} \right) \log n \leq \limsup_{n \rightarrow \infty} \varepsilon_n \log n = 2.$$

Proof of Equation (5). The proof of Equation (5) is useful when the common component enters match utilities non-trivially, i.e. when $\alpha < 1$. In this case, we use a result in Dawande et al., (2001, page 396). Consider a random bipartite graph $G(V_1 \cup V_2, p)$, where $0 < p < 1$ is a constant, $|V_1| = |V_2| = n$, and $\delta_n = \log n / \log \frac{1}{p}$. Let Z_b be the number of bicliques of size $b \times b$. The result shows that

$$Pr(Z_b \geq 1) \leq \frac{1}{(b!)^2}, \quad \text{for every } n > 1.$$

If a maximal balanced biclique of this graph has size $B \times B$, then

$$Pr(B \geq \delta_n) \leq \frac{1}{(\lceil \delta_n \rceil!)^2}, \quad \text{for every } n > 1. \quad (7)$$

For each realization (c, z^f, z^w) , we draw a bipartite graph such that $F \cup W$ is the set of nodes (where F and W constitute the two parts of the graph), and each pair of f_i and w_j

is connected by an edge if and only if at least one of c_{ij} , z_{ij}^f , or z_{ij}^w is lower than or equal to $1 - (1 - \alpha)\varepsilon_n$.

Let

$$\bar{W}(\varepsilon_n; c, z^f, z^w) \equiv \{w_j | \mu^w(j) \in \bar{F}(\varepsilon_n; c, z^f, z^w)\}.$$

Then $\bar{F} \cup \bar{W}$ is a balanced biclique. If a pair (f_i, w_j) from $\bar{F} \cup \bar{W}$ is not connected by an edge, then $c_{ij}, z_{ij}^f, z_{ij}^w > 1 - (1 - \alpha)\varepsilon_n$ implies

$$\begin{aligned} \phi(c_{ij}, z_{ij}^f) &> 1 - (1 - \alpha)\varepsilon_n = \phi(1 - \varepsilon_n, 1), \quad \text{and} \\ \omega(c_{ij}, z_{ij}^w) &> 1 - (1 - \alpha)\varepsilon_n = \omega(1 - \varepsilon_n, 1). \end{aligned}$$

The pair can achieve utilities higher than their utilities under μ^w . This contradicts μ^w being stable.

The approximate upper bound of the sizes of bicliques (7) implies

$$Pr(|\bar{F}(\varepsilon_n; c, z^f, z^w)| \geq \delta_n) \leq \frac{1}{([\delta_n]!)^2}, \quad \text{for every } n > 1,$$

where

$$\delta_n = \frac{\log n}{-\log p_n} \quad \text{and} \quad p_n = 1 - (1 - \alpha)^3 \varepsilon_n^3.$$

Thus,

$$\mathbb{E} \left[\frac{|\bar{F}(\varepsilon_n; c, z^f, z^w)|}{n} \right] \leq \frac{\delta_n}{n} + \frac{1}{([\delta_n]!)^2} \quad \text{for every } n > 1. \quad (8)$$

To prove Equation (5), we need to show $\frac{\delta_n}{n} = o(n^{-1/2})$ and $\frac{1}{([\delta_n]!)^2} = o(n^{-1/2})$ as $n \rightarrow \infty$.

First, as $-\log p_n \geq 1 - p_n$,

$$\frac{\delta_n}{n} = \frac{\log n}{(-\log p_n)n} \leq \frac{\log n}{(1 - p_n)n} = \frac{\log n}{(1 - \alpha)^3 \varepsilon_n^3 n} = \frac{(\log n)^4}{8(1 - \alpha)^3 n} = o(n^{-1/2}).$$

Second, from Stirling's formula,²⁸

$$\frac{1}{(\lceil \delta_n \rceil!)^2} \leq \frac{1}{\left(\sqrt{2\pi\delta_n} (\delta_n/e)^{\delta_n}\right)^2} = \frac{e^{\delta_n}}{2\pi\delta_n^{\delta_n+1}} \quad \text{for every } n > 1.$$

For every sufficiently large n , since $p_n \rightarrow 1$,

$$\delta_n = \frac{\log n}{-\log p_n} > \log n,$$

and, since $\delta_n \rightarrow \infty$,

$$\log \left(\frac{e^{\delta_n}}{2\pi\delta_n^{\delta_n+1}} \right) = -\log(2\pi) - \log \delta_n - \delta_n(\log \delta_n - 1) \leq -\delta_n.$$

Therefore,

$$\frac{1}{(\lceil \delta_n \rceil!)^2} \leq \frac{1}{e^{\delta_n}} \leq \frac{1}{e^{\log n}} = \frac{1}{n} \quad \text{for any sufficiently large } n.$$

Proof of Equation (6). Let $\mu \equiv \{(i, i) | i = 1, \dots, n\}$. By symmetry, each one of the $n!$ matchings has the same probability of being both stable and entailing a sum of firms' idiosyncratic components that is lower than or equal to $(1 - \varepsilon_n)n$. Therefore,

$$P \left(\frac{\sum_{i=1}^n z_{i\mu^w(i)}^f}{n} \leq 1 - \varepsilon_n \right) \leq n! P \left(\mu \text{ is stable and } \frac{\sum_{i=1}^n z_{i\mu(i)}^f}{n} \leq 1 - \varepsilon_n \right).$$

For each realization (c, z^f, z^w) , we consider the following profile of utilities:

$$\begin{aligned} \tilde{u}_{ij}^f &= (1 - \alpha)c_{ij} + \alpha z_{ij}^f & \text{and} & & \tilde{u}_{ij}^w &= (1 - \alpha)c_{ij} + \alpha z_{ij}^w & \text{if } i \neq j, & \text{and} \\ \tilde{u}_{ij}^f &= (1 - \alpha) + \alpha z_{ij}^f & \text{and} & & \tilde{u}_{ij}^w &= (1 - \alpha)c_{ij} + \alpha z_{ij}^w & \text{if } i = j. \end{aligned}$$

Note that $\tilde{u}_{ij}^f > u_{ij}^f$, generically for all $i = j$.

If a pair (f_i, w_j) with $i \neq j$ is a blocking pair of μ under match utilities $(\tilde{u}_{ij}^f, \tilde{u}_{ij}^w)$, then this

²⁸For every $n \geq 1$,

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{r_n} \quad \text{with} \quad \frac{1}{12n+1} \leq r_n \leq \frac{1}{12n}.$$

pair also blocks μ under the actual realized utilities. Thus,

$$P\left(\mu \text{ is stable and } \frac{\sum_{i=1}^n z_{i\mu(i)}^f}{n} \leq 1 - \varepsilon_n\right) \leq P_{\varepsilon_n, n}$$

where $P_{\varepsilon_n, n}$ is the probability that μ is stable with respect to the utilities $(\tilde{u}_{ij}^f, \tilde{u}_{ij}^w)_{i,j}$ and $\frac{\sum_{i=1}^n z_{i\mu(i)}^f}{n} \leq 1 - \varepsilon_n$. We prove that $n!P_{\varepsilon_n, n}$ converges to zero as n increases.

Preparation Steps. We denote by Γ^f the marginal distribution of \tilde{u}_{ij}^f for pairs of (f_i, w_j) who are not matched under μ (i.e., $i \neq j$), and by Γ^w the marginal distribution of \tilde{u}_{ij}^w for any pair of (f_i, w_j) .

We define $\hat{u}_{ij}^f \equiv \Gamma^f(\tilde{u}_{ij}^f)$ and $\hat{u}_{ij}^w \equiv \Gamma^w(\tilde{u}_{ij}^w)$. For each given realization $(\hat{u}_{ii}^f)_{i=1}^n$ and $(\hat{u}_{jj}^w)_{j=1}^n$, the probability that μ is stable with respect to $(\tilde{u}_{ij}^f, \tilde{u}_{ij}^w)_{i,j}$ is

$$\prod_{1 \leq i \neq j \leq n} \left(1 - P[\hat{u}_{ij}^f > \hat{u}_{ii}^f \text{ and } \hat{u}_{ij}^w > \hat{u}_{jj}^w]\right).$$

Note that c_{ij} , z_{ij}^f , and z_{ij}^w are independently and identically distributed, so they are positively associated. Then, for $i \neq j$, the covariance of \hat{u}_{ij}^f and \hat{u}_{ij}^w is non-negative because both $\Gamma^f(\phi(\cdot, \cdot))$ and $\Gamma^w(\omega(\cdot, \cdot))$ are non-decreasing functions of c_{ij} , z_{ij}^f , and z_{ij}^w .

Thus, we have

$$P[\hat{u}_{ij}^f > \hat{u}_{ii}^f \text{ and } \hat{u}_{ij}^w > \hat{u}_{jj}^w] \geq P[\hat{u}_{ij}^f > \hat{u}_{ii}^f]P[\hat{u}_{ij}^w > \hat{u}_{jj}^w] = (1 - \hat{u}_{ii}^f)(1 - \hat{u}_{jj}^w).$$

Since

$$1 - \hat{u}_{ii}^f = 1 - \phi(1, z_{ii}^f) = \alpha(1 - z_{ii}^f),$$

for each realization of c_{ij} , z_{ij}^f , z_{ij}^w for pairs with $i = j$, the probability that μ is stable is bounded above by

$$\prod_{1 \leq i \neq j \leq n} \left(1 - \alpha(1 - z_{ii}^f)(1 - \hat{u}_{jj}^w)\right).$$

We therefore obtain that

$$P_{\varepsilon_n, n} \leq \int \int_{\sum_{i=1}^n z_{ii}^f \leq (1-\varepsilon_n)n} \prod_{1 \leq i \neq j \leq n} \left(1 - \alpha(1 - z_{ii}^f)(1 - \hat{u}_{jj}^w)\right) dz_{\mathbf{ii}}^f d\hat{\mathbf{u}}_{\mathbf{jj}}^w.$$

Now, let $x_i = 1 - z_{ii}^f$ and $y_j = 1 - \hat{u}_{jj}^w$. Then,

$$P_{\varepsilon_n, n} \leq \int_{\substack{\mathbf{0} \leq \mathbf{x}, \mathbf{y} \leq 1 \\ \varepsilon_n n \leq \sum_{i=1}^n x_i}} \prod_{1 \leq i \neq j \leq n} (1 - \alpha x_i y_j) d(\mathbf{x}, \mathbf{y}).$$

Proof of Convergence.

$$\begin{aligned} P_{\varepsilon_n, n} &\leq \int_{\substack{\mathbf{0} \leq \mathbf{x}, \mathbf{y} \leq 1 \\ \varepsilon_n n \leq \sum_{i=1}^n x_i}} \prod_{1 \leq i \neq j \leq n} (1 - \alpha x_i y_j) d(\mathbf{x}, \mathbf{y}) \\ &= \int_{\substack{\mathbf{0} \leq \mathbf{x}, \mathbf{y} \leq 1 \\ \varepsilon_n n \leq \sum_{i=1}^n x_i}} \prod_{1 \leq j \leq n} \underbrace{\left(\int_0^1 \prod_{i \neq j} (1 - \alpha x_i y_j) dy_j \right)}_{(*)} d\mathbf{x}. \end{aligned}$$

We can assume, without loss of generality, that $0 < \alpha < 1/2$ as $(*)$ above decreases in α . Let $t = n^{-7/8}$ and $\delta = e^t(1 - \alpha t)$. Note that $1 - \alpha z \leq \delta e^{-z}$ if $0 \leq z \leq t$. Thus, when $0 \leq y_j \leq t$, we have $0 \leq x_i y_j \leq t$, so $1 - \alpha x_i y_j \leq \delta \exp(-x_i y_j)$. In addition, $1 + \alpha z \leq e^{\alpha z}$ for any z , so $1 - \alpha x_i y_j \leq \exp(-\alpha x_i y_j)$.

Therefore,

$$\begin{aligned} (*) &= \int_0^t \prod_{i \neq j} (1 - \alpha x_i y_j) dy_j + \int_t^1 \prod_{i \neq j} (1 - \alpha x_i y_j) dy_j \\ &= \int_0^t \prod_{i \neq j} \delta \exp(-x_i y_j) dy_j + \int_t^1 \prod_{i \neq j} \exp(-\alpha x_i y_j) dy_j \\ &= \delta \int_0^t \exp\left(-y_j \sum_{i \neq j} x_i\right) dy_j + \int_t^1 \exp\left(-\alpha y_j \sum_{i \neq j} x_i\right) dy_j. \end{aligned}$$

Let

$$s = \sum_{i=1}^n x_i \quad \text{and} \quad s_j = \sum_{i \neq j} x_i.$$

Then,

$$\begin{aligned} (*) &= \delta \int_0^t \exp(-y_j s_j) dy_j + \int_t^1 \exp(-\alpha y_j s_j) dy_j \\ &= \delta \frac{1 - e^{-ts_j}}{s_j} + \frac{e^{-\alpha ts_j} - e^{-\alpha s_j}}{\alpha s_j} \leq \frac{1}{s_j} \left(\delta + \frac{1}{\alpha} \exp(-\alpha ts_j) \right). \end{aligned}$$

We claim that

$$\delta + \frac{1}{\alpha} \exp(-\alpha t s_j) < \exp(n^{-6/7}) \quad (9)$$

for sufficiently large n .

As $s > \varepsilon_n n$, we have $s_j > \varepsilon_n n - 1$. Thus, (9) follows from

$$\exp(n^{-7/8}) + \frac{1}{\alpha} \exp(-\alpha n^{-7/8}(\varepsilon_n n - 1)) < \exp(n^{-6/7}) \quad \text{with sufficiently large } n.$$

Now, we have

$$\begin{aligned} P_{\varepsilon_n, n} &\leq \int_{\substack{\mathbf{0} \leq \mathbf{x} \leq 1 \\ \varepsilon_n n \leq \sum_{i=1}^n x_i}} \Pi_{1 \leq j \leq n} (*) d\mathbf{x} \\ &\leq \int_{\substack{\mathbf{0} \leq \mathbf{x} \leq 1 \\ \varepsilon_n n \leq \sum_{i=1}^n x_i}} \Pi_{1 \leq j \leq n} \left(\frac{1}{s_j} \exp(n^{-6/7}) \right) d\mathbf{x} = \exp(n^{1/7}) \int_{\substack{\mathbf{0} \leq \mathbf{x} \leq 1 \\ \varepsilon_n n \leq \sum_{i=1}^n x_i}} \Pi_{1 \leq j \leq n} \left(\frac{1}{s_j} \right) d\mathbf{x}. \end{aligned}$$

Note that $\left(\log \frac{1}{s_j}\right)' = -\frac{1}{s_j}$. Thus,

$$\sum_{j=1}^n \log \frac{1}{s_j} = \sum_{j=1}^n \left(\log \frac{1}{s} + \log \frac{s}{s_j} \right) = n \log \frac{1}{s} + \sum_{j=1}^n \log \frac{s}{s - x_j}.$$

In the last term,

$$\frac{s}{s - x_j} \leq \frac{s}{s - 1} \leq \frac{\varepsilon_n n}{\varepsilon_n n - 1} \quad \text{for any sufficiently large } n.$$

Thus,

$$\sum_{j=1}^n \log \frac{1}{s_j} \leq n \ln \frac{1}{s} + n \log \frac{\varepsilon_n n}{\varepsilon_n n - 1}.$$

Moreover

$$\varepsilon_n n \log \frac{\varepsilon_n n}{\varepsilon_n n - 1} = \log \left(\left(1 + \frac{1}{\varepsilon_n n - 1} \right)^{\varepsilon_n n} \right) \rightarrow 1, \quad \text{as } n \rightarrow \infty,$$

which implies that

$$\sum_{j=1}^n \log \frac{1}{s_j} \leq n \log \frac{1}{s} + \frac{2}{\varepsilon_n} = n \log \frac{1}{s} + \log n.$$

Therefore,

$$n!P_{\varepsilon_n, n} \leq n! \exp(n^{1/7}) \int_{\varepsilon_n n \leq s} \exp\left(n \log \frac{1}{s} + \log n\right) f_n(s) ds,$$

where $f_n(s)$ is the probability distribution function of s .

From Lemma 1,

$$\begin{aligned} n!P_{\varepsilon_n, n} &\leq n! \exp(n^{1/7}) \int_{\varepsilon_n n \leq s} \exp\left(n \log \frac{1}{s} + \log n\right) \frac{s^{n-1}}{(n-1)!} Pr(r_n \leq s^{-1}) ds \\ &\leq n^2 \exp(n^{1/7}) Pr\left(r_n \leq \frac{1}{\varepsilon_n n}\right) \int_{\varepsilon_n n}^n \frac{1}{s} ds \\ &= n^2 \exp(n^{1/7}) Pr\left(r_n \leq \frac{1}{\varepsilon_n n}\right) (-\log \varepsilon_n) \\ &\leq n^2 \exp(n^{1/7}) Pr\left(r_n \leq \frac{1}{\varepsilon_n n}\right) (\log(\log n)). \end{aligned}$$

Now,

$$\begin{aligned} &n^2 \exp(n^{1/7}) Pr\left(r_n \leq \frac{1}{\varepsilon_n n}\right) (\log(\log n)) \\ &\leq \exp\left(2 \log n + n^{\frac{1}{7}} - n e^{-\frac{1+n^{-\frac{5}{14}}}{\varepsilon_n}}\right) \log(\log n) + o\left(e^{-n^{1/4}}\right) \\ &\leq \exp\left(2 \log n + n^{\frac{1}{7}} - n^{\frac{1}{2} - \frac{n^{-\frac{5}{14}}}{2}}\right) \log(\log n) + o\left(e^{-n^{1/4}}\right) = o\left(\exp(-n^{1/4})\right). \end{aligned}$$

■

7.4. Overall Efficiency – Proof of Proposition 3. 1. As mentioned in the text, Walkup (1979) implies that $E_n \geq 2n - 6$. Since, by definition, $E_n \leq 2n$, combining these observations with the bounds on S_n provided by Proposition 2, the claim follows.

2. We provide a bound on E_n for this environment. For each pair of firm f_i and worker w_j , suppose u_{ij}^f and u_{ij}^w are distributed uniformly on $[0, 1]$. We define $\tilde{u}_{ij} \equiv \frac{u_{ij}^f + u_{ij}^w}{2}$, which has a triangular distribution on $[0, 1]$. We show that

$$E_n = 2 \cdot \max_{\mu \in M} \sum_{i=1}^n \tilde{u}_{i\mu(i)} \geq 2n - 3\sqrt{n} \quad \text{for every } n \geq 2.$$

We consider two random variables \tilde{v}_{ij}^f and \tilde{v}_{ij}^w with cumulative distribution functions

$$H(x) = \begin{cases} 0 & \text{for } 0 \leq x < 1 - 1/\sqrt{2} \\ \sqrt{1 - 2(1-x)^2} & \text{for } 1 - 1/\sqrt{2} \leq x \leq 1. \end{cases}$$

Notice that

$$P\left(\max\{\tilde{v}_{ij}^f, \tilde{v}_{ij}^w\} \leq x\right) = \begin{cases} 0 & \text{for } 0 \leq x < 1 - 1/\sqrt{2} \\ 1 - 2(1-x)^2 & \text{for } 1 - 1/\sqrt{2} \leq x \leq 1 \end{cases}, \text{ and}$$

$$P(\tilde{u}_{ij} \leq x) = \begin{cases} 2x^2 & \text{for } 0 \leq x < 1/2 \\ 1 - 2(2-x)^2 & \text{for } 1/2 \leq x \leq 1 \end{cases}.$$

Therefore,

$$P(\tilde{u}_{ij} \leq x) \leq P(\max\{\tilde{v}_{ij}^f, \tilde{v}_{ij}^w\} \leq x) \quad \text{for } 0 \leq x \leq 1.$$

That is, \tilde{u}_{ij} first order stochastically dominates $\max\{\tilde{v}_{ij}^f, \tilde{v}_{ij}^w\}$.

We denote by $\tilde{v}_{i(k)}^f$ the k 'th highest value of $(\tilde{v}_{ij}^f)_{j=1}^n$. As $H(\cdot)$ is a concave function on the support of the distribution, Jensen's inequality implies that, for any $k = 1, \dots, n$,

$$H\left(\mathbb{E}[\tilde{v}_{i(k)}^f]\right) \geq \mathbb{E}\left[H(\tilde{v}_{i(k)}^f)\right].$$

In addition, $H(\tilde{v}_{i(k)}^f)$ is equal to the k -th highest value of $\{H(\tilde{v}_{ij}^f)\}_{j=1}^n$, and $H(\tilde{v}_{ij}^f)$ is distributed uniformly on $[0, 1]$. Thus,

$$H\left(\mathbb{E}[\tilde{v}_{i(k)}^f]\right) \geq \mathbb{E}\left[H(\tilde{v}_{i(k)}^f)\right] = \frac{n+1-k}{n+1}.$$

Therefore,

$$\mathbb{E}\left[\tilde{v}_{i(k)}^f\right] \geq H^{-1}\left(\frac{n+1-k}{n+1}\right).$$

Identical calculations hold for $\{\tilde{v}_{ij}^w\}_{i=1}^n$ and the corresponding value $\tilde{v}_{j(k)}^w$.

Consider now a random directed bipartite graph with F and W serving as our two classes of nodes, denoted by G . Each firm f_i has arcs to two workers with the highest realized values of \tilde{v}_{ij}^f . Similarly, each worker w_j has arcs to two firms generating the highest realized values of \tilde{v}_{ij}^w .

Let \mathcal{B} denote the set of all directed bipartite graphs containing at least one perfect matching. Let α denote the maximum utilitarian efficiency achievable by matchings in G . We have

$$E_n \geq \mathbb{E}[\alpha | G \in \mathcal{B}] \cdot P(G \in \mathcal{B}).$$

Each pair of firm f_i and worker w_j matched in the utilitarian efficient matching in G has utility \tilde{u}_{ij} which is no less than either $\tilde{v}_{i(2)}^f$ or $\tilde{v}_{j(2)}^w$. Both have expected values no less than $H^{-1}\left(\frac{n+1-k}{n+1}\right)$, which is equal to $1 - \frac{\sqrt{2n}}{n+1}$.

Walkup (1979) illustrates that

$$P(G \in \mathcal{B}) \geq 1 - \frac{1}{5n}.$$

Therefore, we have

$$\begin{aligned} E_n &\geq 2n \cdot \left(1 - \frac{\sqrt{2n}}{n+1}\right) \cdot \left(1 - \frac{1}{5n}\right) \\ &\geq 2n - 3\sqrt{n}. \end{aligned}$$

In the Online Appendix we show that for idiosyncratic preferences, a slightly tighter characterization of the speed of convergence holds. Namely,

$$\lim_{n \rightarrow \infty} \left(1 - \frac{S_n^f}{n}\right) \log n = \lim_{n \rightarrow \infty} \left(1 - \frac{S_n^w}{n}\right) \log n = 1.$$

This, together with the last inequality, completes the claim. ■

7.5. Convergence Speeds with Assortative Preferences – Proof of Proposition 5.

We provide a proof for the case of deterministic common values and assume that

$$(c_1^f, c_2^f, \dots, c_n^f) = (c_1^w, c_2^w, \dots, c_n^w) = \left(\frac{n-1}{n}, \frac{n-2}{n}, \dots, \frac{1}{n}, 0\right).$$

The assumption of deterministic common values is without loss of generality since the distribution of deterministic common values and the empirical distribution of common values from the uniform distribution converge to one another at an exponential rate (see Fact 4 in the

Online Appendix of Lee, 2017).

Let $\varepsilon_n = 6(1 - \beta)n^{-1/4}$ and define

$$B_F(\varepsilon_n; z^f, z^w) \equiv \{f_i \in F \mid u_{i\mu_W}^f \leq (1 - \beta)c_i^f + \beta - (3/5)\varepsilon_n\}.$$

We will use some results in the Online Appendix of Lee (2017) and show that

$$P\left(\frac{|B_F(\varepsilon_n; z^f, z^w)|}{n} > \theta_n\right) \leq \delta_n, \quad (10)$$

with some sequences $\theta_n = O(n^{-1/4})$ and $\delta_n = o(e^{-n^{1/2}})$. Thus,

$$(1 - \beta)\frac{1}{2} + \beta - \frac{S_n^f}{n} \leq \delta_n + (1 - \delta_n)(\theta_n + (1 - \theta_n)(3/5)\varepsilon_n) = O(n^{-1/4}).$$

Identical arguments hold for S_n^w .

Proof of (10). Let F_n denote the set of firms in markets with n participants on each side. We partition the set F_n into $K_n = \lceil n^{1/4} \rceil$ “tiers”.²⁹ For each tier $k = 1, \dots, K_n$, the firms in tier k are given by $F_{k;n}$, where

$$\begin{aligned} F_{k;n} &\equiv \{f_i \in F_n \mid (k-1)n^{3/4} < i \leq kn^{3/4}\} \\ &= \left\{f_i \in F_n \mid 1 - kn^{-1/4} < c_i^f < 1 - (k-1)n^{-1/4}\right\} \quad (\text{as } c_i^f = 1 - \frac{i}{n}). \end{aligned}$$

Define, for any $k = 1, \dots, K_n$,³⁰

$$B_{F_{k;n}}(\varepsilon_n; z^f, z^w) \equiv \{f \in F_{k;n} \mid u_{i\mu_W}^f \leq (1 - \beta)(1 - kn^{-1/4}) + \beta - (2/5)\varepsilon_n\}.$$

If $f_i \in B_F(\varepsilon_n; z^f, z^w)$ and $f_i \in F_{k;n}$, then

$$\begin{aligned} u_{i\mu_W}^f &\leq (1 - \beta)c_i^f + \beta - (3/5)\varepsilon_n \\ &< (1 - \beta)(1 - (k-1)n^{-1/4}) + \beta - (3/5)\varepsilon_n \\ &< (1 - \beta)(1 - kn^{-1/4}) + \beta - (2/5)\varepsilon_n. \end{aligned}$$

²⁹For $x \in \mathbb{R}$, $\lceil x \rceil$ is the smallest integer that is not smaller than x .

³⁰There is a typo in the definition of $B_{F_{k;n}}$ on page 13 of the Online Appendix of Lee (2017): $n^{-1/2}$ should be replaced with $n^{-1/4}$.

Therefore,

$$B_F(\varepsilon_n; z^f, z^w) \subset \bigcup_{k=1}^{K_n} B_{F_{k;n}}(\varepsilon_n; z^f, z^w).$$

Note that

$$\sum_{k=K_n-2}^{K_n} |F_{k;n}| \leq 3n^{3/4}.$$

With arguments similar to those in the Online Appendix of Lee (2017), we can show that

$$P\left(\sum_{k=1}^{K_n-3} |B_{F_{k;n}}| > (K_n - 3)\phi_n\right) \leq 1 - (1 - \psi_n)^{K_n-3},$$

with some sequences $\phi_n = O((\log n)n^{1/2})$ and $\psi_n = o(e^{-\phi_n})$. By taking into account $K_n \geq n^{1/4}$, we obtain

$$P\left(\sum_{k=1}^{K_n} |B_{F_{k;n}}| > \frac{1}{n}(n^{1/4} - 3)\phi_n + 3n^{-1/4}\right) \leq 1 - (1 - \psi_n)^{n^{1/4}-3}.$$

We can then show that

$$\begin{aligned} \theta_n &\equiv \frac{1}{n}(n^{1/4} - 3)\phi_n + 3n^{-1/4} = O(n^{-1/4}), \quad \text{and} \\ \delta_n &\equiv 1 - (1 - \psi_n)^{n^{1/4}-3} = o(e^{-n^{1/2}}). \end{aligned}$$

■

7.6. Severely Imbalanced Markets – Proof of Proposition 6. For any matching μ , let $R_j^w(\mu)$ denote the rank of worker w_j 's partner: $R_j^w(\mu) = 1$ if worker w_j is matched with the most preferred firm, $R_j^w(\mu) = 2$ if worker w_j is matched with the second most preferred firm, etc.

We use Theorem 1 in Ashlagi, Kanoria, and Leshno (2017), which implies that for $0 < \lambda \leq 1/2$,

$$P_n \equiv P\left(\sum_{i=1}^n R_{\mu^f(i)}^w(\mu^f) \geq \frac{n}{-3 \log \lambda}\right)$$

converges to 1 as $n \rightarrow \infty$.

Thus, for $0 < \lambda \leq 1/2$,

$$\begin{aligned}
\frac{S_n^w}{n} &= \frac{\mathbb{E} \left[\sum_{i=1}^n u_{i\mu^f(i)}^w \right]}{n} \\
&= \frac{\mathbb{E}_{\succ} \left[\mathbb{E}_{u|\succ} \left[\sum_{i=1}^n u_{i\mu^f(i)}^w \mid \sum_{i=1}^n R_{\mu^f(i)}^w(\mu^f) \right] \right]}{n} \\
&= \frac{\mathbb{E}_{\succ} \left[\sum_{i=1}^n 1 - \frac{R_{\mu^f(i)}^w(\mu^f)}{n+1} \right]}{n} = 1 - \frac{\mathbb{E} \left[\sum_{i=1}^n R_{\mu^f(i)}^w(\mu^f) \right]}{n+1} \\
&\leq 1 - \frac{nP_n}{-3(n+1)\log\lambda} \rightarrow 1 - \frac{1}{-3\log\lambda} \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

A market with $\lambda > 1/2$ is a market with more workers than those available in a market with $\lambda = 1/2$. Crawford (1991) shows that every worker becomes weakly worse off in μ^f as more workers enter the market, which concludes our proof. \blacksquare

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